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M.Sc.(MATHEMATICS)

III - SEMESTER

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OPTIMIZATION TECHNIQUES

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SYLLABI-BOOK MAPPING TABLE
OPTIMIZATION TECHNIQUES

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UNIT-I NETWORK MODELS

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Structure

1.1 INTRODUCTION

This chapter introduces the concept of minimal spanning tree algorithm with problems. It also consists of definitions of network, connected network, tree and spanning tree with suitable examples.

1.2 Obectives

After going through this unit, you will be able to:

- Define a network
- Define a spanning tree
- Understand the algorithm of minimal spanning tree method
- Solve the problems in minimal spanning tree algorithm.

1.3 Minimal Spanning Tree Algorithms

A network consists of a set of nodes linked by arcs(branches).

Scope and definition of network models:

The multitude of operation research situation can be modeled and solved has networks.

- ➢ Design of offshore natural gas pipeline network connecting the wells (heads) in the gulf of mexico to an inshore delivery point. The objective of the model is to minimize the cost of constructing the pipeline.
- ➢ Determination of the shortest route between two cities in an existing network or roads.
- ➢ Determination of maximum capacity(in tons per year) of a local slurry pipeline network joining coal mines in Wyoming with power plants in Houston.
- ➢ Determination of the time schedule for the activities of a construction project from oil fields to the refineries through a pipeline.
- ➢ Determination of minimum cost flow schedule from oil fields to the refineries through a pipeline network.

The solution of these situation and others like it is accomplished through a variety of network optimization algorithms. This chapter presents the following algorithm.

NOTES

- Minimal spanning tree(Situation 1)

Network :

A network consists of a set of nodes linked by arcs (or branches)

N = { 1,2,3,4,5}

A = {(1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,2), (4,5)}

Flow:

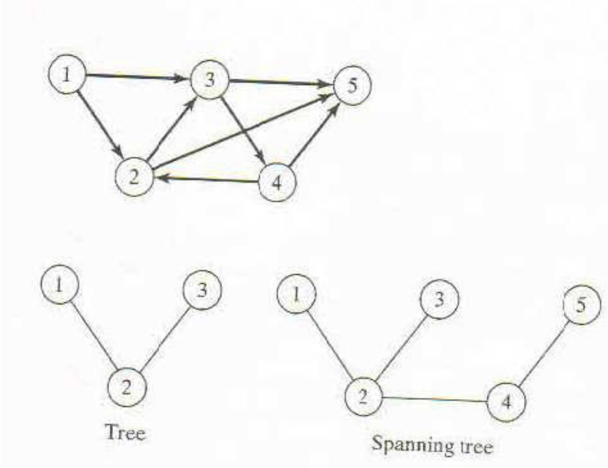
The flow in a network is limited by the capacity of its arcs which may be finite or infinite.

An arc is said to be directed (or) oriented if it allows positive flow in one direction and zero flow in the opposite direction.

A directed network has all directed arcs.

A **path** is a sequence of distinct arcs that join two nodes through other nodes regardless of the direction of flow in each arc.

A **path** forms a cycle (or) a loop if it connects a node to itself through other nodes.



A **connected network** is such that every two distinct nodes linked by atleast one path.

A **tree** is a cycle free connected network comprised of a subset of all the nodes.

A **spanning tree** is a tree that links all the nodes of the network.

Minimal Spanning tree algorithm:

The minimal spanning tree algorithm deals with linking the nodes of a network directly or indirectly. Using the shortest total length of connecting branches. A typical application occurs in the construction of paved roads that link several rural town. The most economical design of the road system calls minimizing the total miles of paved roads.

Algorithm:

Let N = {1,2,…,n} be the set of nodes of the network and define

C_k = Set of nodes that have been permanently connected at iteration k.

$\bar{C}_k$ = Set of nodes or yet to be connected permanently after iteration k.

Step 0:

NOTES

Set $C_0 = \phi$ and $\overline{C_k} = N$

Step 1:

Start with any node i in the unconnected set $\overline{C_0}$ and set $C_1 = \{i\}$ which renders $\overline{C_1} = N - \{i\}$. Set k = 2.

General Step k:

Select a node j* in the unconnected set $\overline{C_{k-1}}$ that yields the shortest arc to a node in the connected set C_{k-1}. Link j* permanently to C_{k-1} and remove it from $\overline{C_{k-1}}$

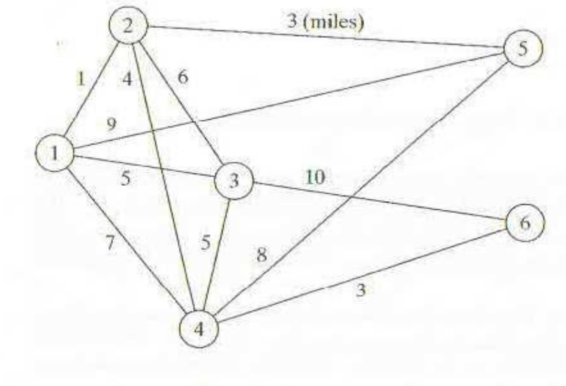
(i.e) $C_k = C_{k-1} + \{j^*\}$, $\overline{C_k} = \overline{C_{k-1}} - \{j^*\}$

If the set of unconnected nodes $\overline{C_k}$ is empty, stop.

Otherwise, set k = k+1 and repeat the step.

Problem:1

The Midwest TV cable company is in the process of providing cable service to 5 new housing development area. The following figure debits possible TV linkages among the 5 areas. The cable miles are shown on each arc. Determine the most economical cable network.



Solution:

Iteration: 0

Set k =0. $C_0 = \phi$, N = {1,2,3,4,5,6} = $\overline{C_0}$

Iteration: 1

Set k = 1. C_1 = {1}, {2,3,4,5,6} = $\overline{C_1}$

Iteration: 2

Set k = 2. C_2= {1,2}, {3,4,5,6} = $\overline{C_2}$

Iteration: 3

Set k = 3. C_3= {1,2,5}, {3,4,6} = $\overline{C_3}$

Iteration: 4

Set k = 4. C_4= {1,2,4,5}, {3,6} = $\overline{C_4}$

Iteration: 5

Set k = 5. C_5= {1,2,4,5,6}, {3} = $\overline{C_5}$

Iteration: 6

Set k = 6. C_6= {1,2,3,4,5.6}, $\overline{C_6} = \phi$.

The resulting network is

Therefore, the most economical cables needed to provide the desired cable service are 1+3+4+5+3 = 16 miles.

NOTES

Problem:2

Solve the minimal spanning tree problem for the network given below

Solution:

Iteration: 0

Set k =0. $C_0 = \phi$, N = {A,B,C,D,E,F,G,H} = $\overline{C}_0$

Iteration: 1

Set k = 1. C_1 = {A}, $\overline{C}_1$ = { B,C,D,E,F,G,H}

Iteration: 2

Set k = 2. C_2= {A,D}, $\overline{C}_2$ = { B,C,E,F,G,H}

Iteration: 3

Set k = 3. C_3= {A,D,E}, $\overline{C}_3$ = { B,C,F,G,H}

Iteration: 4

Set k = 4. C_4= {1,2,4,5}, {3,6} = $\overline{C}_4$

Iteration: 5

Set k = 5. C_5= {1,2,4,5,6}, {3} = $\overline{C}_5$

Iteration: 6

Set k = 6. C_6= {1,2,3,4,5.6}, $\overline{C}_6 = \phi$.

The resulting network is

1.4 Check your progress

1.Define network

2. Define flow

3. Define spanning tree

4. State minimal spanning tree algorithm

1.5 Summary

Network :

A network consists of a set of nodes linked by arcs (or branches)

N = { 1,2,3,4,5}

A = {(1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,2), (4,5)}

Flow:

The flow in a network is limited by the capacity of its arcs which may be finite or infinite.

An **arc** is said to be directed (or) oriented if it allows positive flow in one direction and zero flow in the opposite direction.

A directed network has all directed arcs.

A **path** is a sequence of distinct arcs that join two nodes through other nodes regardless of the direction of flow in each arc.

A **path** forms a cycle (or) a loop if it connects a node to itself through other nodes.

A **connected network** is such that every two distinct nodes linked by atleast one path.

A **tree** is a cycle free connected network comprised of a subset of all the nodes.

A **spanning tree** is a tree that links all the nodes of the network.

NOTES

1.6 Keywords

Flow:
Flow in one direction and zero flow in the opposite direction.
A directed network has all directed arcs.

Path: A path is a sequence of distinct arcs that join two nodes through other nodes regardless of the direction of flow in each arc.

Connected Network: A connected network is such that every two distinct nodes linked by atleast one path.

Tree: A tree is a cycle free connected network comprised of a subset of all the nodes.

1.7 Self Assessment Questions and Exercises

1. Draw the network defined by
   N = {1,2,3,4,5,6}
   A = {(1,2), (1,5),(2,3),(2,4),(3,4),(3,5),(4,3),(4,6),(5,2),(5,6)}
2. Solve the minimum-span problem for the network given below. The numbers on the branches represents the costs of including the branches in the final network.
3. Solve the minimum-span problem for the network given below:
4. Consider eight equal squares arranged in three rows, with two squares in the first row, four in the second, and two in the third. The squares of each row are arranged symmetrically about the vertical axis. It is desired to fill the squares hold consecutive numbers. Use some form of a network representation to find the solution in a systematic way.
5. Seervada-park has recently been set aside for a limited amount of sight-seeing and backpack-hiking. Cars are not allowed into the park. But, there is a narrow winding road system for trams and jeeps driven by the park rangers. This road system is shown (without curves) in the following network, where location O is the entrance into the park; other letters designate the locations of ranger stations (and other limited facilities). The numbers give the distances of these winding roads in miles. Telephone lines , must be installed under the roads to establish telephone communication among all the stations (including the park entrance). Because the installation is both expensive and disruptive to the natural environment, lines will be installed under just-enough roads to provide some connection between every pair of stations. The question is , where the lines should be laid to accomplish this with a minimum total number of milles of lines installed.

1.8Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi,2008.

UNIT–II SHORTEST ROUTE ALGORITHMS

NOTES

Structure

2.1 Introduction

The shortest route problem determines the shortest route between a source and destination in transportation network. Other situations can be represented by the same model.

2.2 Objectives

After going through this unit, you will be able to:

- Define a shortest route problem
- Understand the shortest route algorihms
- Solve the problems by shortest route algorithms

2.3 Shortest Route Algorithms

This section presents two algorithms for solving both cyclic.((i.e) containing loops) and acyclic networks.

- ❖ Dijkstra's Algorithm
- ❖ Floyd's Algorithm

Dijkstra's algorithm is designed to determine the shortest routes between the source node and every other node in the network.

Floyd's algorithm is more general because it allows the determination of the shortest route between any two nodes in the network.

Dijkstra's Algorithm:

Let u_i be the shortest distance from source node 1 to I and define $d_{ij}(\geq 0)$ as the length of arc (I,j). Then, the algorithm defines the label. For an immediately succeeding node j as

$[u_j, i] = [u_i + d_{ij}, i], d_{ij} \geq 0$

The label for the starting node is [0, -] indicating that the node has no predecessor.

Node labels in Dijkstra's algorithm are of two types, **temporary** and **permanent.**

A temporary label is modified if a shorter route to a node can be found. The status of the temporary label is changed to permanent.

Step: 0

Label the source node (node 1) with the permanent label [0,-]. Set i =1.

Step: 1

a) Compute the temporary labels [$u_i + d_{ij}$, i] for each node j that can be reached from node i, provided j is not permanently labeled. If the node j is already labeled with [u_j, k] through another node k and if $u_i + d_{ij} < u_j$, replace [u_j, k] with [$u_i + d_{ij}$, i].

b) If all the nodes have permanent labels, stop. Otherwise select the label [u_r, s] having the shortest distance (= u_r) among all the temporary labels(break ties arbitrarily). Set i = r and repeat step i.

Problem:1

The network in the following figure gives the distances in the miles between pairs of cities 1,2,…,8 use Dikstra's algorithm to find the shortest route between the following cities

a) Cities 1 and 8
b) Cities 1 and 6
c) Cities 4 and 8
d) Cities 2 and 6

Problem:2

Using Dikstra's algorithm to find the shortest route between nodel and every other node in the following network.

Solution:

Iteration: 0

Assign the permanent label [0, -] to node 1.

Iteration: 1

The nodes 2 and 3 can be reached from (the permanent labeled) node 1. Therefore, the list of labeled nodes(permanent) and temporary becomes

Node	Label	Status
1	[0,-]	Permanent
2	[05,1] = [5,1]	Temporary
3	[0+1,1] = [1,1]	Temporary ←

Since min{5,1} = 1, the status of node 3 is changed to permanent.(u_3 =1)

Iteration: 2

The nodes 2,4 and 5 can be reached from the node 3 and the list of labeled nodes becomes

Node	Label	Status
1	[0,-]	Permanent
2	[3,3]	Temporary ←
3	[1,1]	Permanent
4	[1+6,3] = [7,3]	Temporary
5	[1+=7, 3] = [8,3]	Temporary

Since min{3,7,8} = 3, the status of node 2 is changed to permanent.(u_3 = 3)

Iteration:3

The nodes 4,5 and 6 can be reached from the node 2 and the list of labeled nodes becomes

Node	Label	Status
1	[0,-]	Permanent
2	[3,3]	Permanent
3	[1,1]	Permanent
4	[1+6,3] = [7,3]	Temporary
5	[1+2+1,2] = [4,2]	Temporary ←
6	[1+2+6,2] = [9,2]	Temporary

Since min{7,4,9} = 4, the status of node 5 is changed to permanent.(u_5 = 4)

Iteration:4

The nodes 4, 7 and 6 can be reached from the node 5 and the list of labeled nodes becomes

Node	Label	Status
1	[0,-]	Permanent
2	[3,3]	Permanent
3	[1,1]	Permanent
4	[7,3],[7,5]	Temporary ←
5	[4,2]	Permanent
6	[9,2],[9,5]	Temporary
7	[13,5]	Temporary

Since min{7,9,13} = 7, the status of node 4 is changed to permanent.(u_4 = 7)

Iteration:5

The nodes 6 and 7 can be reached from the node 4 and the list of labeled nodes becomes

Node	Label	Status
1	[0,-]	Permanent
2	[3,3]	Permanent
3	[1,1]	Permanent
4	[7,3],[7,5]	Permanent
5	[4,2]	Permanent ←
6	[9,2],[9,5]	Temporary
7	[13,5],[13,4]	Temporary

Since min{9,13} = 9, the status of node 6 is changed to permanent.(u_6 = 9)

Iteration:6

Now the nodes 7 is the only temporary label because node 7 does not lead to any other nodes.

Therefore its status is converted to permanent and the process ends.(u_7 =13).

Shortest Route between node 1 other nodes: [Back tracking method is used here]

Node 1 and 2:

(2) – [3,3] – [1,1] - (1) = 2+1 = 3

Therefore the shortest distance between node 1 and 2 is (1) ⟶ (3) ⟶ (2) with distance 3.

NOTES

Node 1 and 3:

(2) – [1,1] - (1)

Therefore the shortest distance between node 1 and 3 is (1) → (3) with distance 1.

Node 1 and 4:

(4) – [7,3] – [1,1] - (1) = 2+1 = 3

Therefore the shortest distance between node 1 and 4 is (1) → (3) → (4) with distance 7.

(Or)

(4) – [7,5] – [4,2] – [3,3] – [1,1] – (1)

Therefore the distance between node 1 and 4 is (1) → (3) → (2) → (5) → (1) with distance7.

Node 1 and 5:

(5) – [4,2] – [3,3] – [1,1] - (1)

Therefore the shortest distance between node 1 and 4 is (1) → (3) → (2) → (5) with distance 4.

Node 1 and 6:

(6) – [9,2] – [3,3] – [1,1] - (1)

Therefore the shortest distance between node 1 and 4 is (1) → (3) → (2) → (6) with distance 9.

(Or)

(6) – [9,5] – [4,2] – [3,3] – [1,1] – (1)

Therefore the distance between node 1 and 4 is (1) → (3) → (2) → (5) → (6) with distance 9.

Node 1 and 7:

(7) – [13,4] – [7,3] – [1,1] - (1)

Therefore the shortest distance between node 1 and 4 is (1) → (3) → (4) → (7) with distance 13.

(Or)

(7) – [13,4] – [7,5] - [4,2] – [3,3] – [1,1] – (1)

Therefore the distance between node 1 and 4 is (1) → (3) → (2) → (5) → (4) → (7) with distance 13.

(Or)

(7) – [13,5] – [4,2] – [3,3] – [1,1] – (1)

Therefore the distance between node 1 and 4 is (1) → (3) → (2) → (5) → (7) with distance 13.

2.4 Check your progress

1. Define temporary label
2. Define permanent label
3. State Dikstra's algorithm
4. State Floyd's algorithm

2.6 Summary

Dijkstra's algorithm is designed to determine the shortest routes between the source node and every other node in the network.

NOTES

Floyd's algorithm is more general because it allows the determination of the shortest route between any two nodes in the network

Node labels in Dijkstra's algorithm are of two types, **temporary** and **permanent.**

A temporary label is modified if a shorter route to a node can be found. The status of the temporary label is changed to permanent.

2.7 Keywords

Temporary Node: A temporary label is modified if a shorter route to a node can be found.

Permanent Node: The status of the temporary label is changed to permanent.

2.8 Self Assessment Questions and Exercises

1. The network gives the distance in miles between pairs of cities 1,2,… and 8. Use Dijkstra's algorithm to find the shortest route between the following cities.
   a) Cities 1 and 8
   b) Cities 1 and 6
   c) Cities 4 and 8
   d) Cities 2 and 6
2. Use Dijkstra's algorithm to find the shortest route between node 1 and every other node in the network .
3. Use Dijkstra's algorithm to find the shortest route from source 'a' to destination 'f' from the following network.
4. Apply Dijkstra's algorithm to find the shortest route from s to in the graph given below.
5. Use Floyd's algorithm to find the shortest route between each of the following pairs of nodes.
   a) From node 5 to node 1
   b) From node 3 to node 5
   c) From node 5 to node 3
   d) From node 5 to node 2
6. Apply Floyd's algorithm to the network. Arcs (7,6) and (6,4) are unidirectional and all the distance are in miles. Determine the shortest route between the following pairs of nodes.
   a) From node 1 to node 7
   b) From node 7 to node 1
   c) From node 6 to node 7

2.9 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

UNIT-III MAXIMAL FLOW MODELS

Structure

3.1 Introduction

This chapter explains about the importance of maximal flow models in the real life situations like transport of crude oil from refineries, water flow, pipeline of gas etc through maximal flow algorithm and suitable examples.

3.2 Objectives

After going through this unit, you will be able to:

- Define enumeration of cut
- Define cut capacity
- Derive the maximal flow algorithm
- Solve the problems in maximal flow models
- Find the conditions of tangent surfaces

3.3 Maximal Flow Models

Consider a network of pipeline that transports crude oil from oil wells to refineries. Intermediate booster and pumping stations are installed at appropriate design distance to move the crude in the network. Each pipe segment has a finite maximum discharge rate of crude flow (or capacity). A pipe segment may be uni- or bidirectional, depending on its design.

Maximal flow algorithm

The maximal flow algorithm is based on finding breakthrough paths with net positive flow between the source and sink nodes. Each path commits part or all of the capacities of its arcs to the total flow in the network.

Consider arc (i, j) with (initial) capacities $(\overline{C_{ij}}\ ,\ \overline{C_{ji}})$. As portions of these capacities are committed to the flow in the arc, the residuals (or remaining capacities) of the arc are updated. We use the notation $(C_{ij}\ ,\ C_{ji})$ to represent these residuals.

For a node j that receives flow from node i, we attach a label $[a_j, i]$, where a_j is the flow from node i to node j. The steps of the algorithm are thus summarized as follows.

Step 1. For all arcs (i,j), set the residual capacity equal to the initial capacity – that is $(C_{ij}, C_{ji}) = (\overline{C_{ij}}\ ,\ \overline{C_{ji}})$. Let $a_1=\infty$ and label source node 1 with $[\infty, -]$. Set i=1, and go to step 2.

Step 2. Determine S_i, the set of unlabeled nodes j that can be reached directly from nodes i by arcs

With positive residuals (that is $C_{ij} > 0$ for all j £ S_{i}). If $S_i \neq$ □ , go to step 3. Otherwise , goto step 4.

Step 3. Determine k £ S_i such that,

$$C_{ik} = \max_{j\ £\ Si} (C_{ij})$$

NOTES

Set $a_k = C_{ik}$ and label node k with $[a_k , i]$. If k=n the sink node has been labeled , and a breakthrough path is found, go to step 5. Otherwise , set i=k and go to step 2.

Step 4: (**Back tracking**). if i=1, no breakthrough is possible; goto step 6. Otherwise , let r be the nodes that has been labelled immediately before current node i and remove i from the set of nodes adjacent to r. Set i=r and go to step 2.

Step 5. (**Determination of residuals**). Let $N_p = (1, k_1, k_2,......,n)$ define the nodes of the p^{th} breakthrough path from source node 1 to sink node n. Then the maximum flow along path is computed as,

$$F_p = \min(a_1, a_{k1}, a_{k2},.... a_n,)$$

The residual capacity of each arc along the breakthrough path is decreased by f_p in the direction of the flow and increased by f_p in the reverse direction that is, for nodes i and j on the path, the residuals flow is changed from the current (C_{ij} , C_{ji}) to

a. $(C_{ij} - f_p, C_{ji} + f_p)$ if the flow is from i to j.

b. $(C_{ij} + f_p, C_{ji} - f_p)$ if the flow is from j to i.

Reinstate any nodes that were removed in step 4. Set i=1, and return to step 2 to attempt a new breakthrough path.

Step 6. (**Solution**).

a. Given that m break through paths have been determined, the maximal flow in the net work is $F= f_1 + f_2 +........ + f_m$

b. Using the initial and final residuals of arc (i,j), $(\overline{C_{ij}} , \overline{C_{ji}})$ and (C_{ij} , C_{ji}), respectively, the optimal flow in arc (i,j) is computed as follows. Let $(\alpha,\beta) = ((\overline{C_{ij}} - C_{ij} , \overline{C_{ji}} - C_{ji})$. if $\alpha > 0$, the optimal flow from i to j is α. Otherwise , if $\beta > 0$, the optimal flow from j to i is β. (It is impossible to have both α and β positive).

The backtracking process of step 4 is invoked when the algorithm becomes "dead-ended" at an intermediate node. The flow adjustment in step 5 can be explained via the simple flow network. Network (a) gives the first breakthrough path N_1={1,2,3,4} with its maximum flow f_1=5. Thus, the residuals of each of arcs (1,2), (2,3) and (3,4) are changed from (5,0) to (0,5), per step 5. Network (b) now gives the second breakthrough path N_2={1,3,2,4} with f_2=5. After making the necessary flow adjustment, we get network (c), where no further breakthrough are possible. What happened in the transition from (b) to (c) is nothing but a cancellation of a previously committed flow in the direction 2→3. The algorithm is able to "remember" that a flow from 2 to 3 has been committed previously only because we have increased the capacity in the reverse direction from 0 to 5 (per step 5).

1. Determine the maximal flow in the network :

Iteration :1

Set the initial residuals (C_{ij}, C_{ji}) equal to the initial capacities $(\overline{C_{ij}} , \overline{C_{ji}})$.

Step :1

Set $a_1 = \infty$ and label node 1 with $[\infty, -]$. Set i=1.

Step: 2

S_1 ={2,3,4}

Step:3

K=3, because c_{13}=max{ $c_{12,}$ $c_{13,}$ c_{14}} =max{20,30,10}=30. Set a_3= c_{13}=30, and label node 3 with [30,1]. Set i =3, and repeat step 2.

Step:2

S_3 ={4,5}

Step:3

K=5 and a_5= c_{13} =max{10,20}=20. Label node 5 with [20,3]. Breakingthrough is achieved. Go to step 5.

Step:5

The breakthrough path is determined from the labels starting at node 5 and moving backward to node 1 – that is, (5) → [20,3] →(3) →[30,1] →(1). Thus, N_1={1,3,5} and f_1= min{ $a_{1,}$ $a_{3,}$ a_5}={∞, 30,20}=20. The residual capacities along path N_1 are

($c_{13,}$ c_{31})=(30-20, 0+20) =(10,20)

($c_{35,}$ c_{53})= (20-20,0+20) = (0,20)

Iteration :2

Step :1

Set $a_{1=}\infty$ and label node 1 with [∞, −]. Set i=1.

Step: 2

S_1 ={2,3,4}

Step:3

K=2, a_2=c_{12} =max{20,10,10}=20. Set i =2, and repeat step 2.

Step:2

S_2 ={3,5}

Step:3

K=3 and a_3= c_{23}=40. Label node 3 with [40,2]. Set i =3, and repeat step 2.

Step:2

S_3 ={4}

Step:3

K=4 and a_4= c_{34}=10. Label node 4 with [10,3]. Set i =4, and repeat step 2.

Step:2

S_4 ={5}

Step:3

K=5 and a_5= c_{45}=20. Label node 5 with [20,4]. Breakthrough has been achieved. Go to step 5.

Step:5

N_2={1,2,3,4,5} and f_2= {∞, 320,40,10,20}=10. The residual capacities along path N_2 are

($c_{12,}$ c_{21})=(20-10, 0+10) =(10,10)

($c_{23,}$ c_{32})= (40-10,0+10) = (30,10)

($c_{34,}$ c_{43})= (10-10,5+10) = (0,15)

($c_{45,}$ c_{54})= (20-10,0+10) = (10,10)

Iteration :3

Step :1

Set $a_{1=}\infty$ and label node 1 with [∞, −]. Set i=1.

Step: 2

S_1 ={2,3,4}

Step:3

K=2, $a_2=c_{12}$ =max{10,10,10}=10. Label node 2 with [10,1]. Set i =2, and repeat step 2.

Step:2

S_2 ={3,5}

Step:3

K=3 and a_3= c_{23}=30. Label node 3 with [30,2]. Set i =3, and repeat step 2.

Step:2

$S_3 =\emptyset$. Go to step 4 to backtrack.

Step:3 Backtracking:

The label [30,2] at node 3 gives the immediately preceding node r=2. Remove node 3 from further consideration in this iteration by crossing it out. Set i = r = 4, and repeat step 2.

Step:2

S_4 ={5}

Step:3

K=5 and a_5= c_{25}=30. Label node 5 with [30,2]. Breakthrough has been achieved. Go to step 5.

Step:5

N_3={1,2,5} and c_5= min{∞,10,30}=10. The residual capacities along path N_3 are

($c_{12,}$ c_{21})=(10-10, 10+10) =(0,20)

($c_{25,}$ c_{52})= (30-10,0+10) = (20,10)

Iteration:4

This iteration yields N_4={1,3,2,5} with f_4= 10.

Iteration:5

This iteration yields N_5={1,4,5} with f_5= 10.

Iteration:6

All the arcs out of node 1 have zero residuals. Hence, no further breakthrough are possible. We turn to step 6 to determine the solution.

Step:6

Maximal flow in the network is F=f_1+ f_2 +...+f_5 =20+10+10+10+10=60 units. The flow in the different arcs is computed by subtracting the last residuals (c_{ij}, c_{ji}) in iteration 6 from the initial capacities ($\overline{C_{ij}}$, $\overline{C_{ji}}$) , as the following table shows.

Arc	**($\overline{C_{ij}}$, $\overline{C_{ji}}$) - (cij, cji)$_6$**	**Flow amount**	**Direction**
(1,2)	(20,0)-(0,20)=(20,-20)	20	1→2
(1,3)	(30,0)-(0,30)=(30,-30)	30	1→3
(1,4)	(10,0)-(0,10)=(10,-10)	10	1→4
(2,3)	(40,0)-(40,0)=(0,0)	0	-
(2,5)	(30,0)-(10,20)=(20,-20)	20	2→5
(3,4)	(10,5)-(0,15)=(10,-10)	10	3→4
(3,5)	(20,0)-(0,20)=(20,-20)	20	3→5
(4,5)	(20,0)-(0,20)=(20,-20)	20	4→5

Floyd's Algorithm:

Floyd's algorithm is more general than Dijkstra's because it determines the shortest route between any two nodes in the network. The algorithm represents an n-node network as a square matrix with n rows and n columns. Entry (i,j) of the matrix gives the distance d_{ij} from node i to node j, which is finite if i is linked directly to j, and infinite otherwise.

The idea of Floyd's algorithm is straight forward. Given three nodes i,j and k with the connecting distances shown on the three arcs, it is shorter to reach j from i passing through k if $d_{ik}+d_{kj}<d_{ij}$

In this case, it is optimal to replace the direct route from $i \rightarrow j$ with the indirect route $i \rightarrow k \rightarrow j$. This triple operation exchange is applied systematically to the network using the following steps.

Step 0.

Define the starting distance matrix D_0 and node sequence matrix S_0 as given below. The diagonal elements are marked with(-) to indicate that they are blocked. Set k=1.

General Step k.

Define row k and column k as pivot row and pivot column. Apply the triple operation to each element d_{ij} in D_{k-1} for all i and j. If the condition $d_{ik}+d_{kj}<d_{ij}$, ($i \neq k$, $j \neq k$ and $i \neq j$)

is satisfied, make the following changes.

(a) Create D_k by replacing d_{ij} in D_{k-1} with $d_{ik}+d_{kj}$

(b) Create S_k by replacing S_{ij} in S_{k-1} with k. Set k=k+1. If k=n+1, stop; else repeat step k.

Step k of the algorithm can be explained by representing D_{k-1}. Here row k and column k define the current pivot row and column. Row i represent any of the rows 1,2,...., and k-1 and row p represents any of the rows k+1,k+2,... and n. Similarly, column j represents any of the column 1,2,..., and k-1, and column q represents any of the columns k+1, k+2,..., and n. The triple operation can be applied as follows. If the sum of the elements on the pivot row and the pivot column is smaller than the associated intersection element, then it is optimal to replace the intersection distance by the sum of the pivot distances.

After n steps, we can determine the shortest route between node i and j from the matrix D_n and S_n using the following rules.

1. From D_n, d_{ij} gives the shortest distance between nodes i and j.
2. From S_n, determine the intermediate node $k=s_{ij}$ that yields the route $i \rightarrow k \rightarrow j$. If $s_{ik}=k$ and $s_{kj}=j$, stop; all the intermediate nodes of the route have been found. Otherwise, repeat the procedure between nodes i and k, and between nodes k and j.

1. Find the shortest routes between every two nodes. The distance are given on the arcs. Arc(3,5) is directional, so that no traffic is allowed from node 5 to node 3. All the other arcs allow two-way traffic.

   Iteration 0.

   The matrices D_0 and S_0 give the initial representation of the network. D_0 is symmetrical, except that $d_{53}=\infty$ because no traffic is allowed from node 5 to node 3.

NOTES

D_0

-	3	10	∞	∞
3	-	∞	5	∞
10	∞	-	6	15
∞	5	6	-	4
∞	∞	∞	4	-

S_0

-	2	3	4	5
1	-	3	4	5
1	2	-	4	5
1	2	3	-	5
1	2	3	4	-

Iteration 1.

Set k=1. The pivot row and column are shown by the lightly shaded first row and first column in D_0- matrix. The darker cells, d_{23} and d_{32}, are the only ones that can be improved by the triple operation. Thus, D_1 and S_1 are obtained from D_0 and S_0 in the following manner.

1. Replace d_{23} with $d_{21}+d_{13}=3+10=13$ and set $s_{23}=1$.
2. Replace d_{32} with $d_{31}+d_{12}=10+3=13$ and set $s_{23}=1$.

D_1

-	3	10	∞	∞
3	-	13	5	∞
10	13	-	6	15
∞	5	6	-	4
∞	∞	∞	4	-

S_1

-	2	3	4	5
1	-	1	4	5
1	1	-	4	5
1	2	3	-	5
1	2	3	4	-

Iteration 2.

Set k= 2, as shown lightly shaded row and column in D_1. The triple operation is applied to the darker cells in D_1 and S_1. The resulting changes are shown in bold in D_2 and S_2

D_2

-	3	10	8	∞
3	-	13	5	∞
10	13	-	6	15
8	5	6	-	4
∞	∞	∞	4	-

S_2

-	2	3	2	5
1	-	1	4	5
1	1	-	4	5
2	2	3	-	5
1	2	3	4	-

Iteration 3.

Set k= 3, as shown by the shaded row and column in D_2. The new matrices are given by D_3 and S_3.

D_3

-	3	10	8	25
3	-	13	5	28
10	13	-	6	15
8	5	6	-	4
∞	∞	∞	4	1

S_3

-	2	3	2	3
1	-	1	4	3
1	1	-	4	5
2	2	3	-	5
1	2	3	4	-

Iteration 4.

Set k= 4, as shown by the shaded row and column in D_3. The new matrices are given by D_4 and S_4.

D_4

-	3	10	8	12
3	-	11	5	9
10	11	-	6	10
8	5	6	-	4
12	9	10	4	-

S_4

-	2	3	2	4
1	-	4	4	4
1	4	-	4	4
2	2	3	-	5
4	4	4	4	-

Iteration 5.

Set k= 5, as shown by the shaded row and column in D_4. No further improvement are possible in this iteration.

The final matrices D_4 and S_4 contain all the information needed to determine the shortest route between any two nodes in the network.

For example, from D_4 , the shortest distance from node 1 to node 5 is d_{15}=12 miles. To determine the associated route, recall that a segment (i,j) represents a direct link only if S_{ij} =j. Otherwise, i and j are linked through at least one other intermediate node. Because $S_{15} =4 \neq 5$, the route is initially given as $1 \rightarrow 4 \rightarrow 5$. Now because $S_{14} =2 \neq 4$, the segment (1,4) is not direct link, and $1 \rightarrow 4$ is replaced with $1 \rightarrow 2 \rightarrow 4$, and the route $1 \rightarrow 4 \rightarrow 5$ now becomes $1 \rightarrow 2 \rightarrow 4 \rightarrow 5$. Next, because S_{12} =2, S_{24} =4 and S_{45} =5, no further dissecting is needed, and $1 \rightarrow 2 \rightarrow 4 \rightarrow 5$ defines the shortest route.

3.4 Check your progress

1.Define enumeration of cut.
2.Define cut capacity
3. State maximal flow algorithm
4. State floyd's algorthm.

3.5 Summary

The maximal flow algorithm is based on finding breakthrough paths with net positive flow between the source and sink nodes. Each path commits part or all of the capacities of its arcs to the total flow in the network

NOTES

3.6 Keywords

Osculating sphere: The osculating sphere at a point on a curve is the sphere which has four points contact with the curve at p.
Osculating circle: Osculating circle at any point p on a curve is a circle which has three points contact with the curve at the point p. It is also known as circle of curvature.
Tangent surface: A surface of a curve c is called a tangent surface if the surface generated bt tangent to the curve c.
The torsion of the involute : $\tau_1 = \frac{k\tau' - k'\tau}{(a-s)k(\tau^2+k^2)}$

3.7 Self Assessment Questions and Exercises

1. Determine the maximal flow in the network
   a) Determine the surplus capacities for all the arcs,
   b) Determine the amount of flow through nodes 2,3 and 4
   c) Can the network flow be increased by increasing the capacities in the direction 3→ 5 and 4 →5?
2. Determine the maximal flow and the optimum flow in each arc for the network.
3. Suppose that the maximum daily capacity of pump 6 in the network is limited to 50 million bbl per day. Remodel the network to include this restriction. Then determine the maximum capacity of the network.

Suppose that transhipping is allowed between silons 1 and 2 and silos 2 and 3. Suppose also the transhipping is allowed between farms 1 and 2,2 and 3, and 3 and4. The maximum two-way daily capacity on the proposed transhipping routes is 150 (thousand) Ib. What is the effect of transhipping on the unsatisfied demands at the farms?

3.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

UNIT-IV CRITICAL PATH CALCULATIONS

NOTES

Structure

4.1 Introduction

CPM and PERT are network based methods designed to assist in the planning,scheduling and control of the project. A project is defined as the collection of inter-related activities with each activity consuming time and resources. The objective of CPM and PERT is to provide analytic means for scheduling the activities. The two techniques CPM and PERT differ in that CPM assumes deterministic activity duration and PERT assumes probabilities duration.

4.2 Objectives

After going through this unit, you will be able to:

- Define a Critical path
- Identify the rules for the network
- Draw a network
- Calculate the critical path, total and free floats

4.3 Critical Path Calculations

There are 3 rules available for constructing the networks in CPM:

Rule:1

Each activity is represented by one and only arc.

Rule:2

Each activity must be identified by 2 distinct end nodes.

Rule:3

To maintain the correct precedence relationships the following questions must be answered as each activity is added to the network.

a) what activities must immediately proceed the current activity?

b) what activities must follow the current activity?

c) what activities must occur congruently with the current activity?

1. A publisher has a contract with an author to publish a text book. The activities associated with the production of the textbook are given below. The author is require to submit to the publisher a hard copy and a computer file of the menu sprit. Develope the associated network for the project.

Activity	predecessors	duration (weeks)
A: Menu script proof reading by editor	-	3

B: Sample pages preparation	-	2
C:Book cover design	-	4
D: Artwork preparation	-	3
E: Authors approval of editors menuscript and sample pages	A,B	2
F: Book formating	E	4
G: Authors review of formated pages	F	2
H: Authors review of hardwork	D	1
I: Production of printing plates	G,H	2
J: Book production and binding	C,I	4

Solution:

2. Draw a network for job production and indicate the critical path from the following.

Activity	Discription	Time(weeks)	proceeded
A	Market research	15	-
B	Make drawing	15	-
C	Decide product policy	3	A
D	Prepare sales program	5	A
E	Prepare operation seats	8	B,C
F	Buy materials	12	B,C
G	Plane labour force	1	E
H	Make tools	14	E
I	Schedule production	3	D,G
J	Produce product	14	F,H,I

Solution:

Critical Path Computations:

CPM is the construction of time schedule for the project. End of the process we get the following informations.

1) Total duration needed to complete the project.

2) Classification of the activities of the project as critical and non-critical.

Definition:

An activity is said to be critical if there is no leeway in the determining in its start and finish times.

A non-critical activity allows same scheduling slack so that the start time of an activity can be advanced (or) delayed within the limits without affecting the completion date of the entire project.

Define $\square_j$ -earliest occurrence time of the event j.

Δ_j -latest occurrence time of the event j.

D_{ij} -Duration of the activity. (i,j)

The critical path calculations involve two passes.

1)Forward Pass:

It determines the earliest occurrence times of the event.

2) Backward pass:

This calculates their latest occurrence times.

Forward pass: (Earliest occurrence times,$\square$)

The computation start at node 1 and advance to end node n.

Initial step:

Set $\square_1$=0 to indicate that the project start at time 0.

General step j:

Given that node p,q,... and v are linked directly to node j by incoming activities.

(p,j),(q,j),... and (v,j) and that the earliest occurrence times of events p,q,... and v have already been computed then the earliest occurrence time of event j is computed as

$$\square_j = \max[\square_p + D_{pj}, \square + D_{qj}, \ldots \square + D_{vj}]$$

The forward pass is completed when $\square_n$ at node n has been completed.

Backward pass: (Latest occurrence time,$\triangle_j$)

The computations starts at node n and ends at node 1.

Initial step:

Let $\triangle_n = \square_n$ be indicate that the earliest and latest occurrence of node n in the project are same.

General step j:

Given that nodes p,q,... and v are linked directly to node j by outgoing activities (j,p),(j,q),... and (j,v) and that the latest occurrence times of events p,q,... and v have already been computed then the latest occurrence time of event j is computed as

$$\triangle_j = min\{\triangle_p - D_{jp}, \triangle_q - D_{jq}, \ldots \triangle_v - D_{jv}\}$$

The backward pass is completed when $\triangle$ at node 1 has been completed.

At this point $\triangle_1 = \square_1 = 0$

An activity (i,j) will be critical if its satisfies the following 3 condition.

i) $\triangle_i = \square_i$

ii) $\triangle_j = \square_j$

iii) $\triangle_j - \triangle_i = \square_j - \square_i = D_{ij}$

An activity which does not satisfies the above 3 conditions is non-critical.

3. Determine the initial path for the project in the following network.

Solution:

Forward pass:

Node 1:

set $\square_1 = 0$

Node 2:

$$\square_2 = \square_i + D_{12} = 0 + 5 = 5$$

Node 3:

$$\square_3 = max\{\square_1 + D_{13}, \square_2 + D_{23}\}$$

=max{6,5+3=8=8}

Node 4:

$\square_4 = \square_2 + \square_{24} = 5 + 8 = 13$

Node 5:

$\square_5 = max\{\square_3 + D_{35}, \square_4 + D_{45}\}$
=max {8+2,13+0}=13

Node 6:

$\square_6 = max\{\square_3 + D_{36}, \square_4 + D_{46}, \square_5 + D_{56}\}$
=max {8+11,13+1,13+12}
=max {19,14,25}=25

The computations show that the project can be completed in 25 days.

Backward pass:

Node 6:

set $\Delta_6 = \square_6 = 25$

Node 5:

$\Delta_5 = \square_5 - D_{56} = 25 - 12 = 13$

Node 4:

$\Delta_4 = min\{\Delta_6 - D_{46}, \Delta_5 - D_{45}\}$
=min {25-1,13-0}
=min {24,13}=13

Node 3:

$\Delta_3 = min\{\Delta_6 - D_{63}, \Delta_5 - D_{53}\}$
=min {25-11,13-2}=11

Node 2:

$\Delta_2 = min\{\Delta_3 - D_{32}, \Delta_4 - D_{42}\}$
=min {11-3,13-8}=5

Node 1:

$\Delta_1 = min\{\Delta_2 - D_{21}, \Delta_3 - D_{31}\}$
=min {5-5,11-6}=0

The critical path is $1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6$

The sum of the duration of the critical activities [(1,2),(2,4),(4,5)and(5,6)] equals the duration of the project=25 days.

These activities are critical the others are non-critical.

Construction of the time schedule:

In the section we obtain from the calculation of previous section can be used to develope the time schedule for an activity (i,j).$\square_j$ represents by earliest start and Δ_j represent the latest computation time. The interval ($\square_j$,Δ_j) gives the maximum span during which activity (i,j) may be schedule without delaying the entire project.

Determination of the float:

Floats are the slack times available with in the alloted open of the non-critical activities. The most common are the total float free-float.

The total float in the exist of the same spane define from the earliest occurrence of event i to the latest occurrence of event j over the duration of (i,j).This total float denoted by $TF_{ij} = \triangle_j - \square_i - D_{ij}$

The Free-float of time span define from the earliest occurrence of event j over the duration of (i,j)

$FF_{ij} = \square_j - \square_i - D_{ij}$

Note:

$FF_{ij} \leq TF_{ij}$

Red-flagging Rule:

For a non-critical activity (i,j)

a) If $FF_{ij} = TF_{ij}$ then the activity can be scheduled anywhere within its $(\square_j, \triangle_j)$. Span without causing schedule conflict.

b) If $FF_{ij} < TF_{ij}$ then the start of the activity can be delayed by atmost FF_{ij} relative to its earliest start time $\square_i$ without causing schedule conflict.

Any delay larger than FF_{ij} (but not more that TF_{ij}) must be coupled with an equal delay relative to $\square_j$ in the start time of all the activities leaving node j.

4. Compute the floats for the non-critical activities of the network in above problem: 11 and discuss there is finalishing the schedule for the project.

Solution:

Non-critical activity	Duration	Total float(TF) $\triangle_j - \square_i - D_{ij}$	Free float(FF) $\square_j - \square_i - D_{ij}$
B(1,3)	6	5	2
C(2,3)	3	3	0
E(3,5)	2	3	3
F(3,6)	11	6	6
G(4,6)	1	11	11

The computation red-flag activities B and C because their $FF < TF$. The remaining activities (E,F and G) have FF=TF and have may be scheduled anywhere between their earliest start and latest completion times.

Because its FF=2 days, starting B anywhere between time 0 and time 2 will have no effect on the succeeding activities E and F.

If however activities B must start at time $2 + d(\leq 5)$, then the start times of the immediately succeeding activities E and F must be pushed forward past their earliest start time (=8) by atleasted. In this manner, the precedence relationship between B and its successors E and F is perserved.

Turning to red-flag activity C, we note that is FF=0. This means that any delay in starting C past its earliest start time (=5) must be coupled with atleast an equal delay in the start of its successor activities E and F.

4.4 Check your progress

- Define critical and non critical activities
- State Red Flagging rule.

- Define free and total floats.
- State CPM

4.5 Summary

Definition:

An activity is said to be critical if there is no leeway in the determining in its start and finish times.

A non-critical activity allows same scheduling slack so that the start time of an activity can be advanced (or) delayed within the limits without affecting the completion date of the entire project.

Define $\square_j$ -earliest occurrence time of the event j.

$\triangle_j$ -latest occurrence time of the event j.

D_{ij} -Duration of the activity. (i,j)

The critical path calculations involve two passes.

1)Forward Pass:

It determines the earliest occurrence times of the event.

2) Backward pass:

This calculates their latest occurrence times.

Determination of the float:

Floats are the slack times available with in the alloted open of the non-critical activities. The most common are the total float free-float.

The total float in the exist of the same spane define from the earliest occurrence of event i to the latest occurrence of event j over the duration of (i,j).This total float denoted by $TF_{ij} = \triangle_j - \square_i - D_{ij}$

The Free-float of time span define from the earliest occurrence of event j over the duration of (i,j)

$FF_{ij} = \square_j - \square_i - D_{ij}$

Note:

$FF_{ij} \leq TF_{ij}$

Red-flagging Rule:

For a non-critical activity (i,j)

a) If $FF_{ij} = TF_{ij}$ then the activity can be scheduled anywhere within its $(\square_j, \triangle_j)$. Span without causing schedule conflict.

b) If $FF_{ij} < TF_{ij}$ then the start of the activity can be delayed by atmost FF_{ij} relative to its earliest start time $\square_i$ without causing schedule conflict.

Any delay larger than FF_{ij} (but not more that TF_{ij}) must be coupled with an equal delay relative to $\square_j$ in the start time of all the activities leaving node j.

4.6 Keywords

- An activity is said to be critical if there is no leeway in the determining in its start and finish times.
- A non-critical activity allows same scheduling slack so that the start time of an activity can be advanced (or) delayed within the limits without affecting the completion date of the entire project.

- The total float in the exist of the same spane define from the earliest occurrence of event i to the latest occurrence of event j over the duration of (i,j).This total float denoted by $TF_{ij} = \Delta_j - \square_i - D_{ij}$
- The Free-float of time span define from the earliest occurrence of event j over the duration of (i,j)
- $FF_{ij} = \square_j - \square_i - D_{ij}$

4.7 Self Assessment Questions and Exercises

1. Determine the critical path for the project network.
2. Determine the critical path for the project network.
3. Consider the following data for the activities of a project.
4. A project has the following time schedule.

Activity:	1-2	1-3	1-4	2-5	3-6	3-7	4-6	5-8	6-9	7-8	8-9
Duration:	2	2	1	4	8	5	3	1	5	4	3

5. Given an activity (i,j) with duration D_{ij} and its earliest start time $\square_i$ and its latest completion time Δ_j , determine the earliest completion and the latest start times of (i,j).
6. What are the total and free floats of a critical activity? Explain

4.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

BLOCK II: ADVANCEDLINEAR PROGRAMMING AND GAME THEORY

UNIT-V ADVANCED LINEAR PROGRAMMING

Structure

5.1 Introduction

This chapter deals with simplex and generized simplex tableau algorthms with suitable examples under the topic advanced linear programming. The definition of basis, singular and nonsingular matrices, linearly dependent and independent solutions are also explained.

5.2 Objectives

After going through this unit, you will be able to:

- Define basis
- Define singular and non singular matrices.
- Derive the generalized simplex tabuleau algorithm
- Define linearly dependent and independent solutions
- Solve the problems in simplex method.

5.3 ADVANCED LINEAR PROGRAMMING: SIMPLEX METHOD

Convex set:

The feasible solution space is said to form a convex set.If the line segment joining any two distinct feasible points falls in the set.

Extreme point:

An extreme point on the convex set is a feasible point that cannot lie on a line segment joining any two distinct feasible point in this set.

Example:

A feasible point X can be expressed as a convex combination of its extreme points $X_1, X_2, X_3, \dots X_6$ using $X = \alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_6 X_6$, $\alpha_1, \alpha_2, \dots \alpha_6 = 1$, $\alpha_i \geq 0$, i=1,2,...6.

1.Show that the following set is convex $\boldsymbol{C = \{(x_1, x_2)/x_1 \leq 2, x_2 \leq 3, x_1 \geq 0, x_2 \geq 0\}}$

Solution:

Consider $x_1 = \{x_1{}', x_2{}'\}$ and $x_2 = \{x_1{}'', x_2{}''\}$ are distinct points.

If C is convex then $X = (x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ and $\alpha_1 + \alpha_2 = 1$

Now to prove the given set is convex it should satisfies the constraints of C by the line segment X.

(ie) $x_1 = \alpha_1 x_1{}' + \alpha_2 x_1{}'' \leq \alpha_1(2) + \alpha_2(2) = 2$

$x_2 = \alpha_1 x_2{}' + \alpha_2 x_2{}'' \leq \alpha_1(3) + \alpha_2(3) = 3$

Thus $x_1 \leq 2$ and $x_2 \leq 3$ also the non-negativity conditions are satisfy because α_1 and α_2 are non-negative.

2. Show that the set $Q = \{(x_1, x_2 / x_1 \geq 1 or x_2 \geq 2\}$

Solution:

Let $X_1 = \{x_1{}', {}_2{}'\}$ and $X_2 = \{x_1{}', x_2{}'\}$ are two distinct points.

If Q is convex then $X = (x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ and $\alpha_1 + \alpha_2 = 1$

Now to prove: the given set is not convex it should satisfies the condition of C by the line segment X.

$x_1 = \alpha_1 x_1{}' + \alpha_2 x_1{}'' \geq \alpha_1(1) + \alpha_2(1) \geq 1$

$x_2 = \alpha_1 x_2{}' + \alpha_2 x_2{}'' \geq \alpha_1(2) + \alpha_2(2) \geq 2$

Hence $x_1 \geq 1$ or $x_2 \geq 2$

The given set is non-convex.

3. Show that the set $Q = \{x_1 x_2 / x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ is convex.Is the non-negative condition is essential for the proof.

Solution:

Given set $Q = \{x_1 x_2 / x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$

Here given that $x_1 + x_2 \leq 1, x_1 \geq 0, and x_2 \geq 0$.

From this we concluded that the given set is a convex set and also the non-negativity are not essential.

4. Show that the following set is convex $Q = \{x_1 x_2 / x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\}$ extreme point to basic solutions.

Define X as a n-vector representing the variable, A as an (m*n)-matrix representing the constrain coefficient b as column vector representing the right hand side and C as an n-vector representing the objective function coefficients.

Then l.p is written as maximize or minimize Z=CX.

Subject to constraints AX=b, $x \geq 0$

Basic solution:

A basic solution of AX=b is determined by setting n-m variables equal to 0 and then solving the resulting m-equation in the remaining m-unknowns, provided that the resulting solution is unique.

Note:

Extreme points of $\{X / AX = b\} \Leftrightarrow$ basic solution of AX=b.

Basis:

The system AX=b can be expressed in vector form as follows $\sum_{j=1}^{n} p_j x_j = b$, the vector p_j is the j^{th} column of A.

NOTES

A subset of m-vectors is said to form a basis B iff the selected m vectors are linearly independent.

In this case the matrix B is non-singular if X_B is the set of m variable associated with the vector of non-singular B then X_B must be a basic solution. In this case,we have

$$BX_B = b \Rightarrow X_B = B^{-1}b$$

If $B^{-1}b \geq 0$ then X_B is feasible.

In a system of m-equation and n-unknowns the maximum number of feasible and infeasible basic solution is given by

$$\binom{n}{m}=n!/m!\,(n-m)!$$

5. Determine and classify as feasible or infeasible all the basic solutions of the following system of equations.

$$\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-2} & \mathbf{-2} \end{pmatrix} \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{pmatrix} = \begin{pmatrix} \mathbf{4} \\ \mathbf{2} \end{pmatrix}$$

Solution:

Here $P_1 = \binom{1}{2}$; $P_2 = \binom{3}{-2}$; $P_3 = \binom{-1}{-2}$

B	BXB=b	Solution	Status
$\binom{P_1, P_2}{}$	$\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}\binom{x_1}{x_2}=\binom{4}{2}$	$\binom{x_1}{x_2}=\begin{pmatrix} 1/4 & 3/4 \\ 1/4 & -1/8 \end{pmatrix}\binom{4}{2}=\binom{7/4}{3/4}$	feasible
$\binom{P_2, P_3}{}$	$\begin{pmatrix} 3 & -1 \\ -2 & -2 \end{pmatrix}\binom{x_1}{x_2}=\binom{4}{2}$	$\binom{x_2}{x_3}=\begin{pmatrix} 1/4 & -1/8 \\ -1/4 & -3/8 \end{pmatrix}\binom{4}{2}=\binom{3/4}{-7/4}$	infeasible
$\binom{P_1, P_3}{}$	$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\binom{x_1}{x_2}=\binom{4}{2}$	$\binom{x_1}{x_3}=\infty$	not a basis

Consider B=$\binom{P_1 \quad P_2}{}$

Now, $BX_B = b$

$$\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}\binom{x_1}{x_2}=\binom{4}{2}$$

$B_1^{-1} = adjB_1/|B_1|$

Vector representation of the l.p solution space.

Here $\binom{P_1, P_2}{}$ and $\binom{P_2, P_3}{}$ for a basis because its associated vectors are independent.

$\binom{P_2, P_3}{}$ does not form a basis because its associated vectors are dependent.

To calculate the given matrix form a basis or not. we can find the following.

$\det\binom{P_1, P_2}{} = \begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}$

$\det\binom{P_2, P_3}{} = -8 \neq 0$

$\det\binom{P_1, P_3}{} = 0$

from this we conclude that $\binom{P_1, P_2}{}$ and $\binom{P_1, P_3}{}$ form a basis because the determinant values of $\binom{P_1, P_2}{}$ and $\binom{P_1, P_3}{} \neq 0$ and also $\binom{P_1, P_3}{}$ does not form a basis because its determinant value is zero.

6. Determinant and classify all the basis solution of the following system of equation

$\boldsymbol{x_1 + 3x_2 = 2,}$

$\boldsymbol{3x_1 + x_2 = 3.}$

Solution:

$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Here $P_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

$B^{-1} = \frac{adjB}{|B|}$

adj B$= \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$

$|B| = |1 - 9| = |-8| = 8$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/-8 & 3/8 \\ 3/8 & -1/8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7/8 \\ 3/8 \end{pmatrix}$

Thus $\binom{P_1, P_2}{}$ form a basis.

To verify this $\det\binom{P_1, P_2}{} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ =1-9=-8≠0

Graphical form:

7. Consider the following system of equation

$\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix} \boldsymbol{x_1} + \begin{pmatrix} \mathbf{0} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix} \boldsymbol{x_2} + \begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{2} \end{pmatrix} \boldsymbol{x_3} + \begin{pmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \boldsymbol{x_4} = \begin{pmatrix} \mathbf{3} \\ \mathbf{4} \\ \mathbf{2} \end{pmatrix}$

Determine if any of the following combinations forms a basis.

a) $\boldsymbol{\binom{P_1, P_2, p_3}{}}$

b) $\boldsymbol{\binom{P_1, P_2, p_4}{}}$

c) $\boldsymbol{\binom{P_2, P_3, p_4}{}}$

d) $\boldsymbol{\binom{P_1, P_2, p_3, p_4}{}}$

NOTES

Simplex method:
Simplex algorithm:

Step 1:
Check whether the objective function is to be maximized or minimized. If it is to be minimized then convert into a maximization problem by minimize Z=-max(-Z)
Step 2:
Check whether all the bi's are positive. If any one of the bi's are negative multiply both sides of the constraints by (-1) so as to make its right hand side positive.
Step 3:
By introducing slack, surplus variables convert the inequality constraints into equations and express the LPP in standard form.
Step 4:
Find an Initial basic feasible solution (IBFS) and express the information in simplex table.
Step 5:
Compute a net evaluate $Z_j - C_j$

i) If all $Z_j - C_j \geq 0$ then the current basic feasible solution X_B is optimal.

ii) If atleast one of the $Z_j - C_j < 0$ then the current basic feasible solution is not optimal.
Step 6:
Finding the entering variable
The entering variable is the non-basic variable corresponding to the most negative value of $Z_j - C_j$. The entering variables column is known as key column (or) pivot column
If more than one variable has the same most negative $Z_j - C_j$ then any one of the variable may be selected arbitrarily as the entering variable.
Step 7:
Finding the leaving variable
Compute the ratio $\theta = min(X_B/X_i), X_i > 0$

i) If all $X_i \leq 0$ then there is unbounded solution to the given CPP.

ii) If atleast one $X_i > 0$ then the leaving variable is the basic variable corresponding to the minimum ratio θ The leaving variable ρ is known as key row or pivot row and the intersection of pivot row and pivot column gives the pivot element.
Step 8:
Drop the leaving variable and introduce the entering variable along with its associated value under C_B columns convert the pivot elements to unity and all others elements in its column to zero by making use are

i) New pivot equation =old pivot equation/pivot element

ii) new all other equation including $Z_j - C_j$ row

= old equation-(corresponding column coefficient)*(new pivot equation).

NOTES

Step 9:

Repeat the above procedure until either an optimum solution is obtained.

8. Use simplex method to solve the LPP maximize Z=$4x_1 + 10x_2$ subject to the constraints $2x_1 + x_2 \leq 50$,

$2x_1 + 5x_2 \leq 100$,

$2x_1 + 3x_2 \leq 90$ and $x_1, x_2 \geq 0$.

Solution:

Introduce the slack variables S_1, S_2, S_3 and convert the inequalities into equation.

max Z=$4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to, $2x_1 + x_2 + s_1 = 50$

$2x_1 + 5x_2 + s_2 = 100$

$2x_1 + 3x_2 + s_3 = 90$

$x_1, x_2, s_1, s_2, s_3 \geq 0$ (standard form)

Initial table:

		c_j		0				Ratio
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	50	2	1	1	0	0	50/1=50
0	s_2	100	2	5	0	1	0	**100/5=20**
0	s_3	90	2	3	0	0	1	90/3=30
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		-4	**-10**	0	0	0	

Iteration 1:

		c_j	4	1 0	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	30	8/5	0	1	-1/5	0
0	x_2	20	2/5	1	0	1/5	0
0	s_3	30	4/5	0	0	-3/5	1
	Z_j	200	4	10	0	2	0
	$Z_j - C_j$		0	**0**	0	2	0

Since all $Z_j - C_j \geq 0$

The optimum solution is obtained

NOTES

$x_1 = 0, x_2 = 0,$

max Z=200

9. Find the non-negative values x_1, x_2, x_3 which maximizes Z=$3x_1 + 2x_2 + 5x_3$ subject to the constraints $x_1 + 4x_2 \leq 420$

$$\mathbf{3x_1 + 2x_3 \leq 460}$$

$$\mathbf{x_1 + 2x_2 + x_3 \leq 430.}$$

Solution:

Introduce the slack variable s_1, s_2, s_3 and convert the inequalities into equation.

max Z=$3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to, $x_1 + 4x_2 + s_1 = 420$

$3x_1 + 2x_3 + s_2 = 460$

$x_1 + 2x_2 + x_3 + s_3 = 430$

Initial table:

		c_j	3			(		(	Ratio
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	420	1	4	0	1	0	0	420/0=∞
0	s_2	460	3	0	2	0	1	0	**460/2=230**
0	s_3	430	1	2	1	0	0	1	430/1=430
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		3	-2	**-5**	0	0	0	

Iteration 1:

		c_j	3						Ratio
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	420	1	4	0	1	0	0	420/4=105
5	x_3	230	3/2	0	1	0	1/2	0	230/0=∞
0	s_3	200	-1/2	**2**	0	0	-1/2	1	**200/2=100**
	Z_j	1150	15/2	0	5	0	5/2	0	
	$Z_j - C_j$		9/2	-2	0	0	5/2	0	

Iteration 2:

		c_j	3					**0**
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	20	2	0	0	1	0	-2
5	x_3	230	3/2	0	1	0	1/2	0
2	x_2	200	-1/4	1	0	0	-1/4	1/2

	Z_j	1350	14/2	2	5	0	4/2	1
	$Z_j - C_j$		4	0	0	0	4/2	1

NOTES

Since all $Z_j - C_j \geq 0$

The optimum solution is obtained

$x_1 = 0, x_2 = 100, x_3 = 230$

$maxZ = 1350$

Generalized simplex tableau in matrix form:

Consider the LPP in equation form maximize Z=CX subject to the constraints AX=B, $X \geq 0$

This equation can be rewritten as follows $\begin{pmatrix} 1 & -C \\ 0 & A \end{pmatrix} \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$

Suppose B is a feasible basis of the system AX=B, $X \geq 0$ and let X be the corresponding vector of basic variables with C_B as its associated objective vector.

Then the corresponding solution can be written as

$$\begin{pmatrix} Z \\ X_B \end{pmatrix} = \begin{pmatrix} 1 & -C_B \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} C_B B^{-1} b \\ B^{-1} b \end{pmatrix}$$

Then the general simplex tableau in matrix form can be derived from the original standard equation as follows.

$$\begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 1 & -C \\ 0 & A \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 & C_B B^{-1} - C \\ 0 & B^{-1} A \end{pmatrix} \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} C_B B^{-1} b \\ B^{-1} b \end{pmatrix}$$

Given P_j is the j^{th} vector of A then the simplex tableau column associated with the variable x_j can be represented as

Basis x_j Solution

Z $C_B B^{-1} P_j - C_j$ $C_B B^{-1} b$

X_B $B^{-1} P_j$ $B^{-1} b$

10. Consider the following LP maximize Z=$x_1 + 4x_2 + 7x_3 + 5x_4$ subject to

$$\mathbf{2x_1 + x_2 + 2x_3 + 4x_4 = 10}$$

$$\mathbf{3x_1 - x_2 - 2x_3 + 6x_4 = 5, x_1, x_2, x_3, x_4 \geq 0}$$

Generate the simplex tableau associated with the basis $B = (p_1, p_2)$

Solution:

Given $B = (p_1, p_2) = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$

$X_B = (x_1, x_2)^T = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$C_B = (1,4)$

NOTES

$C_j = \begin{pmatrix} 1,4,7,5 \end{pmatrix}$

$$b = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 10,5 \end{pmatrix}^T$$

Now, $B^{-1} = \frac{adjB}{|B|}$

$$adjB = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} = -2-3 = -5$$

$$|B| = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} = -2-3 = -5$$

$$B^{-1} = \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix}$$

Then we get, $X_B = B^{-1}b$

$$= \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix} \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

To compute the constrain columns in the body simplex tableau we have

$$B^{-1}P_j = B^{-1} \begin{pmatrix} p_1, p_2, p_3, p_4 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 & 4 \\ 3 & -1 & -2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

Now we compute the objective rows as follows,

$$C_B B^{-1} P_j = \begin{pmatrix} 1,4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 8 & 2 \end{pmatrix}$$

$$C_B B^{-1} P_j - C_j = \begin{pmatrix} 1 & 4 & 8 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 7 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & -3 \end{pmatrix}$$

Now $C_B B^{-1} b = C_B X_B$

$$= \begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = (3+16)$$

Z=19

Hence the entire simplex tableau can be summarize as follows

Basis	x_j	**Solution**
Z	0 0 1 -3	19
x_1	1 0 0 2	3
x_2	0 1 2 0	4

11. Consider the following LP maximize Z=$5x_1 + 12x_2 + 4x_3$ subject to

$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$2x_1 - 2x_2 - x_3 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Check if each of the following matrices forms a (feasible or infeasible) basis: $\begin{pmatrix} p_1, p_2 \end{pmatrix} \begin{pmatrix} p_2, p_3 \end{pmatrix} \begin{pmatrix} p_3, p_4 \end{pmatrix}$

12. In the following LP compute the entire simplex tableau associated with $X_B = \begin{pmatrix} x_1, x_2, x_5 \end{pmatrix}^T$

NOTES

minimize Z=$2x_1 + x_2$ subject to

$$3x_1 + x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_1 + 2x_2 = x_5 = 3, x_1, x_2, x_3, x_4, x_5 \geq 0$$

Solution:

Given $X_B = \begin{pmatrix} x_1, x_2, x_5 \end{pmatrix}^T$

B=$\begin{pmatrix} p_1, p_2, p_5 \end{pmatrix}$

$$P_j = \begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 4 & 3 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$C_j = \begin{pmatrix} -2 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$C_B = \begin{pmatrix} -2 & -1 & 0 \end{pmatrix}$$

b=$\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$=$\begin{pmatrix} 3 & 6 & 3 \end{pmatrix}^T$

B=$\begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

$|B| = 3(3) - 1(4) + 0$ =9-4 =5

$$B^{-1} = \frac{adjB}{|B|}$$

adj B=$\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$

$B_{11} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$=3

$B_{12} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$=1 $B_{13} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$=0

$B_{21} = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$=4 $B_{22} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$=3

$B_{23} = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$=0 $B_{31} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$=5

$B_{32} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$=5 $B_{33} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$=5

adj B=$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 3 & -5 \\ 0 & 0 & 5 \end{pmatrix}$

$$B^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$B^{-1}P_j = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 4 & 3 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -3/5 & 1/5 & 0 \\ 0 & 1 & 4/5 & -3/5 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$C_B B^{-1} P_j = \begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3/5 & 1/5 & 0 \\ 0 & 1 & 4/5 & -3/5 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & 6/5 - 4/5 & -2/5 + 3/5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & 2/5 & 1/5 & 0 \end{pmatrix}$$

$$C_B B^{-1} P_j - C_j = \begin{pmatrix} -2 & -1 & 2/5 & 1/5 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2/5 & 1/5 & 0 \end{pmatrix}$$

$$C_B B^{-1} b = C_B X_B$$

$$= \begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3/5 & 6/5 & 0 \end{pmatrix} = (-6/5-6/5+0) = -12/5$$

Basis	x_j	**Solution**
Z	0 0 2/5 1/5 0	-12/5
x_1	1 0 -3/5 1/5 0	3/5
x_2	0 1 4/5 -3/5 0	6/5
x_5	0 0 -1 1 1	0

13. $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{-2} & \mathbf{-1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{pmatrix} = \begin{pmatrix} \mathbf{10} \\ \mathbf{2} \end{pmatrix}$

Here $P_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $P_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$P_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $P_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

B	**BXB=b**	**Solution**	**Status**
(P_1, P_2)	$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$	$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/6 & -2/6 \\ -2/6 & 1/6 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$	(feasible)
(P_2, P_3)	$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$	$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1/0 & -1/0 \\ 2/0 & 2/0 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \infty$	(not a basis)
(P_3, P_4)	$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$	$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 12 \end{pmatrix}$	(infeasible)

det $(P_1, P_2) = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = -2-4 = -6 \neq 0$

det $(P_2, P_3) = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = -2+2 = 0$

det $(P_3, P_4) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = 0+1 = 1 \neq 0$

(P_1, P_2) and (P_3, P_4) form a basis.

(P_2, P_3) does not form a basis.

5.4 Check your progress

1. Define basis
2. Define singular and non singular matrices.
3. Derive the generalized simplex tabuleau algorithm
4. Define linearly dependent and independent solutions
5. Solve the problems in simplex method

NOTES

5.5 Summary

Basic solution:

A basic solution of AX=b is determined by setting n-m variables equal to 0 and then solving the resulting m-equation in the remaining m-unknowns, provided that the resulting solution is unique.

Note:

Extreme points of $\{X/AX = b\} \Leftrightarrow$ basic solution of AX=b.

Basis:

The system AX=b can be expressed in vector form as follows $\sum_{j=1}^{n} p_j x_j = b$, the vector p_j is the j^{th} column of A.

A subset of m-vectors is said to form a basis B iff the selected m vectors are linearly independent.

In this case the matrix B is non-singular if X_B is the set of m variable associated with the vector of non-singular B then X_B must be a basic solution. In this case,we have

$$BX_B = b \Rightarrow X_B = B^{-1}b$$

If $B^{-1}b \geq 0$ then X_B is feasible.

In a system of m-equation and n-unknowns the maximum number of feasible and infeasible basic solution is given by

$\binom{n}{m}$=$n!/m!\,(n-m)!$

5.6 Keywords

Basic solution:

A basic solution of AX=b is determined by setting n-m variables equal to 0 and then solving the resulting m-equation in the remaining m-unknowns, provided that the resulting solution is unique.

Basis:

The system AX=b can be expressed in vector form as follows $\sum_{j=1}^{n} p_j x_j = b$, the vector p_j is the j^{th} column of A.

Linearly independent:A subset of m-vectors is said to form a basis B iff the selected m vectors are linearly independent.

Singular and Nonsingular matrices: In this case the matrix B is non-singular if X_B is the set of m variable associated with the vector of non-singular B then X_B must be a basic solution. In this case,we have

$$BX_B = b \Rightarrow X_B = B^{-1}b$$

If $B^{-1}b \geq 0$ then X_B is feasible.

NOTES

5.7 Self Assessment Questions and Exercises

1. In the following sets of equations, (a) and (b) have unique solutions, (c) has an infinity of solutions, and (d) has no solution. Show how these results can be verified using graphical vector representation. From this state the general conditions for vector dependence- independence that lead to unique solution, infinity of solutions and no solution.

   a) $x_1+3x_2=2$, $3x_1+x_2=3$
   b) $2x_1+3x_2=1$, $2x_1-x_2=2$
   c) $2x_1+6x_2=4$, $x_1+3x_2=2$
   d) $2x_1-4x_2=2$, $-x_1+2x_2=1$

2. Consider the following system of equations:
   $\begin{pmatrix}1\\2\\3\end{pmatrix}x_1+\begin{pmatrix}0\\2\\1\end{pmatrix}x_2+\begin{pmatrix}1\\4\\2\end{pmatrix}x_3+\begin{pmatrix}2\\0\\0\end{pmatrix}x_4=\begin{pmatrix}3\\4\\2\end{pmatrix}$
   Determine if any of the following combination forms a basis.
   a) (P_1,P_2,P_3)
   b) (P_1,P_2,P_4)
   c) (P_2,P_3,P_4)
   d) (P_1,P_2,P_3,P_4)

3. Consider the following system of equations:

$$\begin{pmatrix}1\\2\\3\end{pmatrix}x_1+\begin{pmatrix}0\\2\\1\end{pmatrix}x_2+\begin{pmatrix}1\\4\\2\end{pmatrix}x_3+\begin{pmatrix}2\\0\\0\end{pmatrix}x_4=\begin{pmatrix}3\\4\\2\end{pmatrix}$$

Consider B=(P_3, P_4) show that the corresponding basic solution is feasible, then general the corresponding simplex tableau.

4. Consider the following LP:

Maximize z = $5x_1+12x_2+4x_3$
Subject to $x_1x_1+2x_2+x_3+x_4=10$
$2x_1-2x_2-x_3=2$
$x_1, x_2, x_3, x_4 \geq 0$
Check if each the following matrices forms a basis: $(P_1, P_2)(P_2, P_3)(P_3, P_4)$.

5. In the following LP, compute the entire simplex tableau associated with $X_B=(x_1,x_2,x_5)^T$

Minimize z= $2x_1+x_2$
Subject to $3x_1+x_2-x_3=3$
$4x_1+3x_2-x_4=6$
$x_1+2x_2+x_5=3$
$x_1,x_{2,}x_{3,}x_{4,}x_5 \geq 0$

5.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

UNIT- VI REVISED SIMPLEX AND BOUNDED VARIABLES ALGORITHMS

Structure

6.1 Introduction

Having developed the optimality and feasibility conditions in chapter 5,the computational steps of the revised simplex method are presented here. In LP models, variables may have explicit positive upper and lower bounds. For example, in production facilities, lower and upper bounds can represent the minimum and maximum demands for certain products. Bounded variables also arise prominently in the course of solving integer programming problems by the branch-and-bound algorithm. The bounded algorithm is efficient computationally because it accounts for the bounds implicity. The problems in revised simplex method and bounded variable algorithm are given here.

6.2 Objectives

After going through this unit, you will be able to:

- Define a revised simplex algorithm
- Define a bounded variable algorithm
- Solve the problems in revised simplex method and bouded variables method
- Give an example in bouded variable algorithm
-

6.3 Curves on surfaces

Revised Simplex Method:

The optimum solution of a linear program is always associated with a basic solution. The simplex method search for the optimum starts by selecting a feasible basis B, and then moving to anothers basis, Bnext, that yields a better (or, atleast, no worse) value of the objection function.

Continuing in this manner, the optimum basis is eventually reached.

The iterative steps of the revised simplex method are exactly the same as in the tableau simplex method. The main difference is that the computations in the revised method are the based on matrix manipulation rather than on row operations. In the tableau simplex method, each tableau is generated from the immediately preceding one which tends to the problem rounoff error.

Development of the optimality and feasibility conditions:

The general LP problem can be written as follows,

maximize (or) minimize Z=$\sum_{j=1}^{n} c_j x_j$

subject to, $\sum_{j=1}^{n} p_j x_j = b$, $x_j \geq 0$, j=1,2,...n

For a given basic vector X_B and its corresponding basis B and objective vector C_B the general simplex tableau developed in previous section shows that any simplex iteration can be represented by the following equations.

Z+$\sum_{j=1}^{n} (Z_j - c_j)x_j$=$C_B B^{-1} b$

$(X_B)_i$+$\sum_{j=1}^{n} (B^{-1}p_j)_i x_j = (B^{-1}b)_i$

$Z_j - C_j$ the reduced cost is defined as $Z_j - C_j$=$C_B B^{-1} P_j - C_j$

The notation $(v)_i$ is used to represent the i^{th} element of the vector v.

Optimality Condition:

From the Z-equation given above an increase in non-basic x_j above its current zero value will improve the value of Z relative to its current value $(= C_B B^{-1} b)$ only if its $Z_j - C_j$ is strictly negative in the case of maximization and strictly positive in the case of minimization. Otherwise x_j cannot improve the solution and must remain non-basic at zero level.

Though any non-basic variable satisfying the given condition can be choosen to improve the solution the simplex method uses a rule of thumb that calls for selecting the entering variable as the one with the most negative (most positive) $Z_j - C_j$ in case of maximization(minimization).

Feasibility Condition:

The determination of the leaving vector is based on examining the constraint equation associated with the type i^{th} basic variable specifically we have,

$(X_B)_i$+$\sum_{j=1}^{n} (B^{-1}p_j)_i x_j = (B^{-1}b)_i$

when the vector P_j is selected by the optimality condition to enter the basis its associated variable x_j will increase above zero level. At the same time, all the remaining non-basic variables remain at zero level. Thus the i^{th} constraint equation reduces to

$(X_B)_i$=$(B^{-1}b)_i - (B^{-1}p_j)_i x_j$

The equation show that $(B^{-1}p_j)_i > 0$ an increase in x_j can cause $(X_B)_i$ to become negative which violates the non-negatively condition $(X_B)_i \geq 0$ for all i

Thus we have $(B^{-1}b)_i - (B^{-1}p_j)_i x_j \geq 0$ for all i.

The condition yields the maximum value of the entering variable x_j as,

$$x_j = min\{\frac{(B^{-1}b)_i}{(B^{-1}P_j)_i} / (B^{-1}P_j)_i > 0\}$$

The basic variable responsible for producing the minimum ratio leaves the basis solution to become non-basic at zero level.

Revised Simplex Algorithm:

Step 0:

Construct a starting basic feasible solution and let B and C_B be its associated basis and objective coefficients vector respectively.

Step 1:

Compute the inverse B^{-1} by using an appropriate inversion method

Step 2:

For each non-basic variable x_j compute $Z_j - C_j = C_B B^{-1} P_j - C_j$

If $Z_j - C_j \geq 0$ in maximization ($\leq 0 minimization$).

For all non-basic x_j stop, the optimal solution is given by $X_B = B^{-1}b, Z = C_B X_B$.

Else, apply the optimality and determine the entering variable x_j as the non-basic variable with the most negative (+ve).

$Z_j - C_j$ incase of maximization (minimization).

Step 3:

Compute $B^{-1}P_j$, if all the elements of $B^{-1}P_j$ are negative (or) zero. Stop the problem has no bounded solution.

Else compute $B^{-1}P$ then for all the strictly positive elements of $B^{-1}P_j$ determine the ratios defined by the feasibility condition. The basic variable x_j associated with the smallest ratio is the leaving variable.

Step 4:

From the current basic B from a new basis by replacing the leaving vector P_i with the entering vector P_j go to step 1 to starts a new iteration.

1.Reddy mikks produces both interior and exterior points from 2-raw materials M_1 and M_2. The following provides the basic data of the problem

	Tons of raw material per tin of		**maximum daily availability**
	Ex.point	**Int.point**	**(tons)**
Raw material M_1	**6**	**4**	**24**
Raw material M_2	**1**	**2**	**6**
profit/ton(rs.1000)	**5**	**4**	**10**

A market survey indicate that the daily demand for interior point cannot exceed that for exterior paint by more than one tons. Also the maximum daily demand for interior paint is 2-times. Ready mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

Solution:

The given LP can be rewritten as max Z=$5x_1 + 4x_2$

Subject to constraints,

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$x_2 - x_1 \leq 1, x_2 \leq 2, x_1, x_2 \geq 0$

The standard form of the given LP is

maximize Z=$5x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to, $6x_1 + 4x_2 + x_3 = 24$

$x_1 + 2x_2 + x_4 = 6$

$x_2 - x_1 + x_5 = 1$

$x_2 + x_6 = 2$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

The above problem can be expressed in matrix form as follows.

maximize Z=$\begin{pmatrix}5,4,0,0,0\end{pmatrix}.\begin{pmatrix}x_1, x_2, x_3, x_4, x_5, x_6^T\end{pmatrix}$

Subject to,

$$\begin{pmatrix}6 & 4 & 1 & 0 & 0 & 0\\ 1 & 2 & 0 & 1 & 0 & 0\\ -1 & 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{pmatrix}=\begin{pmatrix}24\\ 6\\ 1\\ 2\end{pmatrix}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

C=$(c_1, c_2, c_3, c_4, c_5, c_6)$ and $(p_1, p_2, p_3, p_4, p_5, p_6)$ represent the column vectors of the constraints equation and b represent the right hand side value of constraints.

Iteration 0:

$X_{B0} = \begin{pmatrix}x_3, x_4, x_5, x_6\end{pmatrix}^T$

$C_{B0} = \begin{pmatrix}0,0,0,0\end{pmatrix}$

$B0 = \begin{pmatrix}p_3, p_4, p_5, p_6\end{pmatrix}$=I

$B0^{-1} = I$

Thus $X_{B0} = B0^{-1}b$

$$X_{B0} = I\begin{pmatrix}24\\ 6\\ 1\\ 2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}24\\ 6\\ 1\\ 2\end{pmatrix}$$

$X_{B0} = \begin{pmatrix}24,6,1,2\end{pmatrix}^T$

$$Z = C_{B0}X_{B0} = \begin{pmatrix}0,0,0,0\end{pmatrix}\begin{pmatrix}24\\ 6\\ 1\\ 2\end{pmatrix}$$

Z=0

Optimality condition:

$C_{B0}B0^{-1} = \begin{pmatrix}0,0,0,0\end{pmatrix}$ I $\begin{pmatrix}0,0,0,0\end{pmatrix}$

$\{Z_j - C_j\}_{j=1,2}$=$C_{B0}B0^{-1}\begin{pmatrix}p_1, p_2\end{pmatrix}$ - $\begin{pmatrix}c_1, c_2\end{pmatrix}$

=$\begin{pmatrix}0,0,0,0\end{pmatrix}$- $\begin{pmatrix}5,4\end{pmatrix}$ = $\begin{pmatrix}-5, -4\end{pmatrix}$

P_1 is the entering vector.

Feasibility Condition:

We know that $X_{B0} = \begin{pmatrix} x_3, x_4, x_5, x_6 \end{pmatrix}^T$

$=\begin{pmatrix} 24,6,1,2 \end{pmatrix}^T$

$$B0^{-1}P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$B0^{-1}P_1 = \begin{pmatrix} 6,1,-1,0 \end{pmatrix}^T$

$x_1 = min\{24/6, 6/1, 1/-1, 2/0\} = min\{4, 6, -1, \infty\}$

$x_1 = min\{4, 6, -, -\}$

P_3 is the leaving vector.

The result can be written as follows

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	-5	4	0	0	0	0	0
x_3	6						24
x_4	1						6
x_5	-1						1
x_6	0						2

Iteration 1:

$X_{B1} = \begin{pmatrix} x_1, x_4, x_5, x_6 \end{pmatrix}^T$

$C_{B1} = \begin{pmatrix} 5,0,0,0 \end{pmatrix}$

$$B1 = \begin{pmatrix} p_1, p_4, p_5, p_6 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$B^{-1} = \frac{adjB}{|B|}$

$$B_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1(1-0)-0=1$$

$$B_{12} = -\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1(1-0)=-1$$

$$B_{13} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1(0-0)-1(-1+0)+0=1$$

$$B_{14} = -\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 1(0)-1(0)+0=0$$

$$B_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$B_{22} = -\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6(1-0)+0=-6$$

$$B_{23} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$B_{24} = -\begin{pmatrix} 6 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$B_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$B_{32} = -\begin{pmatrix} 6 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$B_{33} = \begin{pmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6(1)+0=6$$

$$B_{34} = -\begin{pmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$B_{41} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0$$

$$B_{42} = -\begin{pmatrix} 6 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} = 0$$

$$B_{43} = \begin{pmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 0$$

$$B_{44} = -\begin{pmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = 6(1)=6$$

$$adjB_1 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -6 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$B_1 = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} -0+0=6(1(1-0))=6$$

$$B_1^{-1} = \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $X_{B1} = B1^{-1}b = \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \\ 2 \end{pmatrix}$

$Z = C_{B1}.X_{B1}$

$= \begin{pmatrix} 5 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \\ 2 \end{pmatrix}$

Z=20

Optimality Condition:

$C_{B1}B1^{-1} = \begin{pmatrix} 5 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 5/6 & 0 & 0 & 0 \end{pmatrix}$

$(Z_j - C_j)_{j=2,3} = C_{B1}B1^{-1}\begin{pmatrix} P_2, P_3 \end{pmatrix} - \begin{pmatrix} C_2, C_3 \end{pmatrix}$

$= \begin{pmatrix} 5/6 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \end{pmatrix}$

$= \begin{pmatrix} 10/3 & 5/6 \end{pmatrix} - \begin{pmatrix} 4 & 0 \end{pmatrix} = \begin{pmatrix} -2/3 & 5/6 \end{pmatrix}$

Then P_2 is the entering vector.

Feasibility condition:

$X_{B1} = \begin{pmatrix} x_1, x_4, x_5, x_6 \end{pmatrix}^T = \begin{pmatrix} 4 & 2 & 5 & 2 \end{pmatrix}^T$

$B1^{-1}P_2 = \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2/3 \\ -2/3 + 2 \\ 2/3 + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 4/3 \\ 5/3 \\ 1 \end{pmatrix}$

Thus $x_2 = min\{\frac{4}{2/3}, \frac{2}{4/3}, \frac{5}{5/3}, \frac{2}{1}\} = min\{6, 3/2, 3, 2\}$

$x_2 = 3/2$

Then P_4 is the leaving vector.

Iteration 2:

$X_{B2} = \begin{pmatrix} x_1, x_2, x_5, x_6 \end{pmatrix}$

$C_{B2} = \begin{pmatrix} 5 & 4 & 0 & 0 \end{pmatrix}$

$$B_2 = (p_1, p_2, p_5, p_6) = \begin{pmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Take $a_{33} = \begin{pmatrix} 6 & 4 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 8$

Cofactor:

$$a_{11} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2,$$

$$a_{12} = -\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

$$a_{13} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 3$$

$$a_{14} = -\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 1$$

$$B2^{-1} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

Thus $X_{B2} = B2^{-1}b$

$$= \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix}$$

$$X_{B2} = \begin{pmatrix} 3 & 3/2 & 5/2 & 1/2 \end{pmatrix}^T$$

$$Z = C_{B2} X_{B2}$$

$$= \begin{pmatrix} 5 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{pmatrix} = 15+6$$

$$Z = 21$$

Optimality condition:

$$C_{B2} B2^{-1} = \begin{pmatrix} 5 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/2 & 0 & 0 \end{pmatrix}$$

$$(Z_j C_j)_{j=3,4} = C_{B2}.B2^{-1}(p_3, p_4) - (c_3, c_4)$$

$$=\begin{pmatrix}3/4 & 1/2 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}3/4 & 1/2\end{pmatrix}-\begin{pmatrix}0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}3/4 & 1/2\end{pmatrix}$$

Since all $Z_j - C_j \geq 0$,

The optimal solution,

$$x_1 = 3, x_2 = 3/2, maxZ = 21$$

2. max Z=$x_1 + x_2$ subject to

$$\mathbf{3x_1 + 2x_2 \leq 6}$$

$$\mathbf{x_1 + 4x_2 \leq 4, x_1, x_2 \geq 0}$$

Solution:

The matrix form is

$$\text{max Z}=\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1 & x_2 & x_3 & x_4\end{pmatrix}^T$$

$$\begin{pmatrix}3 & 2 & 1 & 0\\1 & 4 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}6\\4\end{pmatrix}$$

We use the notation

C=$\begin{pmatrix}c_1 & c_2 & c_3 & c_4\end{pmatrix}$ to represents the objective function coefficient and $\begin{pmatrix}p_1 & p_2 & p_3 & p_4\end{pmatrix}$ to represent the column vector of the constraint equation the R.H.S of the constraint given the given b

Iteration:0

$$X_{B0} = \begin{pmatrix}x_3, x_4\end{pmatrix}^T$$

$$C_{B0} = \begin{pmatrix}c_3, c_4\end{pmatrix}=\begin{pmatrix}0,0\end{pmatrix}$$

$$B0 = \begin{pmatrix}p_3, p_4\end{pmatrix}=I$$

$$B0^{-1} = I$$

Next, $X_{B0} = B0^{-1}b = I.\begin{pmatrix}6,4\end{pmatrix}^T=\begin{pmatrix}6,4\end{pmatrix}^T$

$$Z=C_{B0}X_{B0} = 0$$

Optimality Condition:

$$C_{B0}B^{-1}=\begin{pmatrix}0,0\end{pmatrix}$$

$$(Z_j - C_j)_{j=1,2} = C_{B0}.B0^{-1}\begin{pmatrix}p_1 & p_2\end{pmatrix} - \begin{pmatrix}c_1 & c_2\end{pmatrix} = 0-(1,1)=(-1,-1)$$

Then p_1 is the entering vector.

Feasibility Condition:

$$X_{B0} = \begin{pmatrix}x_3 & x_4\end{pmatrix}^T$$

$X_{B0} = \begin{pmatrix} 6 & 4 \end{pmatrix}^T$

$B0^{-1}p_1 = I.\begin{pmatrix} 3 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \end{pmatrix}^T$

Thus $x_1 = min\{6/3, 4/1\}$=min(2,4)

Then p_3 is the leaving vector.

Iteration 1:

$X_{B1} = \begin{pmatrix} x_1 & x_4 \end{pmatrix}$

$C_{B1} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} p_1 & p_4 \end{pmatrix}$

Thus $B_1 = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$

$B1^{-1} = 1/3 \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}$

Next, $X_{B1} = B1^{-1}b$

$= \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$X_{B1} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \end{pmatrix}^T$

Z=$C_{B1}X_{B1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \end{pmatrix}^T$

Z=2

Optimality Condition:

$C_{B1}B1^{-1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \end{pmatrix}$

$(Z_j - C_j)_{j=2,3} = C_{B1}B1^{-1} \begin{pmatrix} p_2, p_3 \end{pmatrix} - \begin{pmatrix} c_2, c_3 \end{pmatrix}$

$= \begin{pmatrix} 1/3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2/3 & 1/3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -1/3 & 1/3 \end{pmatrix}$

Then p_2 is the entering variable.

Feasibility Computation:

$X_{B1} = \begin{pmatrix} x_1 & x_4 \end{pmatrix}^T = \begin{pmatrix} 2 & 2 \end{pmatrix}^T$

$B1^{-1}p_2 = \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 2/3 \\ -2/3 + 4 \end{pmatrix}$

$B1^{-1}p_2 = \begin{pmatrix} 2/3 \\ 10/3 \end{pmatrix} = \begin{pmatrix} 2/3 & 10/3 \end{pmatrix}^T$

Thus $x_2 = min\{\frac{2}{2/3}, \frac{2}{10/3}\}$ =min(3,3/5)

Then p_4 is the leaving vector.

Iteration 2:

$X_{B2} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$, $C_{B2} = \begin{pmatrix} 1,1 \end{pmatrix}$

$B2 = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

$B2^{-1} = 1/10 \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{pmatrix}$

Next, $X_{B2} = B2^{-1}b$

$= \begin{pmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 12/5 & -4/5 \\ -6/10 & 12/10 \end{pmatrix}$

$X_{B2} = \begin{pmatrix} 8/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 8/5 & 3/5 \end{pmatrix}^T$

$Z = C_{B2}X_{B2} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 8/5 & 3/5 \end{pmatrix}^T$ =(8/5+3/5)

$Z = 11/5$

Optimality Condition:

$C_{B2}B2^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{pmatrix} = \begin{pmatrix} 3/10 & 1/10 \end{pmatrix}$

$(Z_j - C_j)_{j=3,4} = C_{B2}B2^{-1} \begin{pmatrix} p_3 & p_4 \end{pmatrix} - \begin{pmatrix} c_3 & c_4 \end{pmatrix}$

$= \begin{pmatrix} 3/10 & 1/10 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/10 & 1/10 \end{pmatrix}$

Since $Z_j - C_j \geq 0$

We obtain the optimal solution

$x_1 = 8/5 x_2 = 3/5 max Z = 11/5$

3. max Z=$3x_1 + 2x_2 + 5x_3$ subject to

$\boldsymbol{x_1 + 2x_2 + x_3 \leq 430}$

$\boldsymbol{3x_1 + 2x_3 \leq 460}$

$\boldsymbol{x_1 + 4x_2 \leq 420}$ **and** $\boldsymbol{x_1, x_2, x_3 \geq 0}$

Solution:

The matrix form is

max Z=$\begin{pmatrix} 3 & 2 & 5 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}^T$

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 430 \\ 460 \\ 420 \end{pmatrix}$$

C=$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}$ to represent to objective function coefficient and $\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{pmatrix}$ to represent the column vector of the constraints equation

Iteration 0:

$X_{B0} = \begin{pmatrix} x_4 & x_5 & x_6 \end{pmatrix}$

$C_{B0} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$

$B0 = \begin{pmatrix} p_4 & p_5 & p_6 \end{pmatrix}$

$B0 = I; B0^{-1} = I$

Next $X_{B0} = B0^{-1}b$ =I$\begin{pmatrix} 430 & 460 & 420 \end{pmatrix}^T$ =$\begin{pmatrix} 430 & 460 & 420 \end{pmatrix}$

$Z = C_{B0}X_{B0} = 0$

Optimality Condition:

$C_{B0}B0^{-1} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$

$(Z_j - C_j)_{j=1,2,3} = C_{B0}B0^{-1}\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$

=0-$\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ =$\begin{pmatrix} -3 & -2 & -5 \end{pmatrix}$

Thus p_3 is the entering vector.

Feasibility Condition:

$X_{B0} = \begin{pmatrix} x_4 & x_5 & x_6 \end{pmatrix}^T$ =$\begin{pmatrix} 430 & 460 & 420 \end{pmatrix}^T$

$B0^{-1}p_3 = I\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T$ =$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T$

Thus $x_3 = min\left\{\frac{430}{1}, \frac{460}{2}, \frac{420}{0}\right\}$ =min{$430,230,\infty$}=230

p_5 is the leaving vector.

Iteration 1:

$X_{B3} = \begin{pmatrix} x_1 & x_2 & x_6 \end{pmatrix}$

$C_{B3} =_{B0}= \begin{pmatrix} 0 & 5 & 0 \end{pmatrix}$

$B3 =_{B0}= \begin{pmatrix} p_4 & p_3 & p_6 \end{pmatrix}$

Thus $B3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$=2, $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$=0, $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$=0,

-$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$=-1, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$=1, -$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$=0,

$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$=0, -$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$=0, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$=2

adj $B3$=$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^T$

$|B3|$=$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$=1(2-0)-0+0=2

$B3^{-1} = \frac{1}{|B3|}.adjB3$

=1/2.$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^T$ =$\begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$ =$\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Next, $X_{B3} = B3^{-1}b$

$$=\begin{pmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}430\\460\\420\end{pmatrix}=\begin{pmatrix}200\\230\\420\end{pmatrix}$$

$$X_{B3}=\begin{pmatrix}200 & 230 & 430\end{pmatrix}^T$$

$$Z=C_{B3}X_{B3}$$

$$=\begin{pmatrix}0 & 5 & 0\end{pmatrix}\begin{pmatrix}200 & 230 & 430\end{pmatrix}^T=0+1150+0$$

$$Z=1150$$

Optimality Condition:

$$C_{B3}B3^{-1}=\begin{pmatrix}0 & 5 & 0\end{pmatrix}\begin{pmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\0 & 0 & 1\end{pmatrix}$$

$$(Z_j-C_j)_{j=2,3,4}=C_{B3}B3^{-1}\begin{pmatrix}p_2 & p_3 & p_4\end{pmatrix}-\begin{pmatrix}c_2 & c_3 & c_4\end{pmatrix}$$

$$=\begin{pmatrix}0 & 5/2 & 0\end{pmatrix}\begin{pmatrix}2 & 1 & 1\\0 & 2 & 0\\4 & 0 & 0\end{pmatrix}-\begin{pmatrix}2 & 5 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 5 & 0\end{pmatrix}-\begin{pmatrix}2 & 5 & 0\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\end{pmatrix}$$

P_2 is the entering vector.

Feasibility Condition:

$$X_{B3}=\begin{pmatrix}x_4 & x_5 & x_6\end{pmatrix}^T=\begin{pmatrix}200 & 230 & 420\end{pmatrix}^T$$

$$B3^{-1}P_2=\begin{pmatrix}1 & -1/2 & 0\\0 & 1/2 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\0\\4\end{pmatrix}=\begin{pmatrix}2\\0\\4\end{pmatrix}$$

$$B3^{-1}P_2=\begin{pmatrix}2 & 0 & 4\end{pmatrix}^T$$

Thus $x_2=min\{\frac{200}{2},\frac{230}{0},\frac{420}{4}\}=min(100,\infty,105)=100$

p_4 leaves the basis

Iteration 2:

$$X_{B2}=\begin{pmatrix}x_2 & x_3 & x_6\end{pmatrix}^T$$

$$C_{B2}=\begin{pmatrix}2 & 5 & 0\end{pmatrix}$$

$$B2=\begin{pmatrix}p_2 & p_3 & p_6\end{pmatrix}$$

Thus $B2=\begin{pmatrix}2 & 1 & 0\\0 & 2 & 0\\4 & 0 & 1\end{pmatrix}$

$$\text{adj } B2=\begin{pmatrix}2 & 0 & -8\\-1 & 2 & 4\\0 & 0 & 4\end{pmatrix}^T=\begin{pmatrix}2 & -1 & 0\\0 & 2 & 0\\-8 & 4 & 4\end{pmatrix}$$

$$|B2|=\begin{pmatrix}2 & 1 & 0\\0 & 2 & 0\\4 & 0 & 1\end{pmatrix}=2(2)=4$$

NOTES

$$B2^{-1} = 1/4\begin{pmatrix}2 & -1 & 0\\ 0 & 2 & 0\\ -8 & 4 & 4\end{pmatrix} = \begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$$

$$X_{B2} = B2^{-1}b$$

$$= \begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}430\\ 460\\ 420\end{pmatrix} = \begin{pmatrix}100\\ 230\\ 20\end{pmatrix}$$

$$X_{B2} = \begin{pmatrix}100 & 230 & 20\end{pmatrix}^T$$

$$Z = C_{B2}X_{B2} = \begin{pmatrix}2 & 5 & 0\end{pmatrix}\begin{pmatrix}100 & 230 & 20\end{pmatrix}^T = 200+1150=1350$$

Optimality Condition:

$$C_{B2}.B2^{-1} = \begin{pmatrix}2 & 5 & 0\end{pmatrix}\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2 & 0\end{pmatrix}$$

$$(Z_j - C_j)_{j=3,4,5} = C_{B2}B2^{-1}\begin{pmatrix}p_3 & p_4 & p_5\end{pmatrix} - \begin{pmatrix}c_3 & c_4 & c_5\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 2 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}5 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}5 & 1 & 2\end{pmatrix} - \begin{pmatrix}5 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 2\end{pmatrix}$$

Since all $Z_j - C_j \geq 0$,

We obtain the optimal solution.

(ie) $x_3 = 230, x_2 = 100, x_1 = 0, Z = 1350$

Bounded variables algorithm:

The bounded algorithm is efficient computationally, because its accounts for the bounds implicitly.

We consider the lower bounds first because it is simplex.

Bounded variable Technique:

In general, the variables of a LPP upto satisfy the non-negativity constraints. (ie)$x_j \geq 0$

In certain LPP some or all of the variables are have lower and upper limit to the values such a problem can be written as

max Z=CX subject to

AX=b

$l_j \leq x \leq u_j, (j = 1,2,\ldots n)$

where l_i and u_j are lower and upper limit of the variables x_j.

We make convert these constraints to equality by introducing slack and surplus variables as follows.

$x_j + x_j' = u_j$

$$x_j - x_j' = l_j$$
$$x_j = l_j + x_j'$$

We use the special technique for upper bound constraints.

In the regular simplex method all non-basic variables are assign zero values. But in the upper bound technique are basic variables Big-M non-basic with its upper bound variables.

Also when the non-basic variables enter the basis is should not exist its upper bound.

Procedure of bounded variable technique:

Step 1:

Write the given LPP in the standard form by introducing slack and surplus variables.

Step 2:

In any variables x_j has lower bound its should be substituted with $x_j = l_j + x_j{}'$ where $x_j{}' \geq 0$.

Step 3:

Obtain an initial basic solution.

Step 4:

If $Z_j - C_j \geq 0$ for all j then the current solution is optimal. Otherwise choose the most negative $Z_j - C_j$ then the corresponding non-basic variables x_r enters the basis.

Step 5:

Compute the quantities

$$\theta_1 = min\{\frac{X_{Bi}}{a_{ir}}, a_{ir} > 0\}$$
$$\theta_2 = min\{\frac{u_i - X_{Bi}}{-a_{ir}}, a_{ir} < 0\}$$
$\theta_3 = u_r$ [u_r is the upper bound for the entering variable]
$$\theta = min\{\theta_1, \theta_2, \theta_3\}$$

Note:

$a_{ir} \geq 0$, then $\theta_2 = \infty$

Step 6:

If $\theta = \theta_1$, corresponding to X_{Bk}/a_k then X_k is leaves the basis and regular row operation can be performed.

If $\theta = \theta_2$ then the corresponding X_k leaves the basis, x_j becomes non-basic at its upper bound. substituted $x_k = u_k - x_k{}'$. Introduce x_r and drop x_k and perform row operation.

$$(X_{Bi})' = X_{Bi} - a_{ik} u_k \text{ and}$$
$$Z_0{}' = Z_0 - (Z_k - C_k) u_k$$

If $\theta = u_r$ then x_r remains non-basic.

Step 7:

Go to Step 4 and repeat the procedure till the optimality to reached.

4. Use bounded variable technique and solved max Z=$x_1 + 3x_2 - 2x_3$

subject to $x_2 - 2x_3 \leq 1$

$2x_1 + x_2 + 2x_3 \leq 8$

$0 \leq x_1 \leq 1, 0 \leq x_2 \leq 3, 0 \leq x_3 \leq$

Solution:

The problem becomes

max Z=$x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2$

subject to $x_2 - 2x_3 + s_1 = 1$

$2x_1 + x_2 + 2x_3 + s_2 = 8$

$u_1 = 1, u_2 = 3, u_3 = 2, s_1 = \infty, s_2 = \infty$

Initial table:

		c_j	1		-2			Ratio
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	
0	s_1	1	0	**1**	-2	1	0	**1/1=1**
0	s_2	8	2	1	2	0	1	8/1=8
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$	0	-1	**-3**	2	0	0	
	u_j	-	1	3	2	∞	∞	

x_2 enter the basis $\theta_1 = min\{1/1, 8/1\}$=1 corresponding to s,

$\theta_2 = \infty; \theta_3 = u_2 = 3$

Then $\theta = min\{\theta_1, \theta_2, \theta_3\}$ =min$\{1, \infty, 3\}$=1

1 corresponds to S_1.

S_1 leaves the basis

Iteration 1:

		c_j	1	3	-2	0	**0**
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2
3	x_2	1	0	1	**-2**	1	0
0	s_2	7	2	0	4	-1	1
	Z_j	3	0	3	-6	3	0
	$Z_j - C_j$	3	-1	0	-4	3	0
	u_j	-	1	3	2	∞	∞

x_3 enter the basis

$\theta_1 = min\{-, 7/4\}$=7/4

$\theta_2 = min\{\frac{u_i - X_{Bi}}{-a_{ir}}, a_{ir} < 0\}$

$\theta_2 = min\{\frac{u_2 - X_{B1}}{-a_{13}}\}$=1

$\theta_3 = u_3 = 2$

Then $\theta = min\{7/4, 1, 2\}$=1

1 corresponds to x_2

x_2 leaves the basis

-2 is the pivot element

NOTES

Iteration 2:

		c_j		3	-2	0	**0**	Ratio
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	
-2	x_3	-1/2	0	-1/2	1	-1/2	0	**-**
0	s_2	9	2	2	0	1	1	9/2
	$Z_j - C_j$	1	-1	**-2**	0	1	0	
	u_j	-	1	3	2	∞	∞	

Since X_2 is non-basic it must be substituted as its upper bound by units

$x_2 = u_2 - x_2' = 3 - x_2'$

$X_{B1}' = X_{B1} - a_{12}u_2$=-1/2+3/2=1

$X_{B2}' = X_{B2} - a_{22}u_2$=9-2(3)=3

$Z_0' = Z_0 - (Z_2 - C_2)u_2$ =1-(-2)3=1+6=7

Thus the updated simplex table is $x_2' = -x_2$

		c_j	1	-3	-2	0	**0**
C_B	Y_B	X_B	x_1	x ’	x_3	s_1	s_2
-2	x_3	1	0	½	1	-1/2	0
0	s_2	3	2	½	0	1	1
	$Z_j - C_j$	7	-1	2	0	1	0
	u_j	-	1	3	2	∞	∞

x_1 enters

$\theta_1 = min\{1/0, 3/2\}$=3/2

$\theta_2 = \infty$

$\theta_3 = u_1 = 1$

$\theta = min\{\theta_1, \theta_2, \theta_3\}$

=min$\{3/2, \infty, 1\}$=1

1 corresponds to u_1

x_1 is substituted at its upper bound by using

$x_1 = u_1 - x_1' = 1 - x_1'$

$(X_{B1})' = X_{B1} - a_{11}u_1 = 1 - 0(1) = 1$

$(X_{B2})' = X_{B2} - a_{21}u_2$ =3-2(1)=1

$Z_0' = Z_0 - (Z_1 - C_1)u_1$=7-(-1)(1)=8

		c_j	-1	-3	-2	0	**0**
C_B	Y_B	X_B	,	, x_2	x_3	s_1	s_2
-2	x_3	1	0	½	1	-1/2	0
0	s_2	1	-2	-2	0	1	1
	$Z_j - C_j$	8	1	2	0	1	0
	u_j	-	1	3	2	∞	∞

Since all $Z_j - C_j \geq 0$

Hence the optimal solution is attained

$x_3 = 1, x_1{}' = 0$

$x_2{}' = 0$

$x_1 = 1 - x_1{}' = 1 - 0 = 1$

$x_2 = 3 - 0 = 3$

$x_1 = 1$

$x_2 = 3$

$x_3 = 1$

max Z=8

4. Solve the following LP model by the upper bounding algorithm.

max Z=$3x_1 + 5y + 2x_3$ subject to

$\boldsymbol{x_1 + y + 2x_3 \leq 14}$

$\boldsymbol{2x_1 + 4y + 3x_3 \leq 43}$

$\boldsymbol{0 \leq x_1 \leq 4, 7 \leq y \leq 10, 0 \leq x_3 \leq 3}$

Solution:

Let $y = x_2 + 7$

$7 \leq x_2 + 7 \leq 10$

$0 \leq x_2 \leq 3$

The given LP becomes

max Z=$3x_1 + 5x_2 + 2x_3 + 0s_1 + 0s_2$

Subject to $x_1 + x_2 + 2x_3 + s_1 = 7$

$2x_1 + 4x_2 + 3x_3 + s_2 = 15$

$0 \leq x_1 \leq 4$, $0 \leq x_2 \leq 3$, $0 \leq x_3 \leq 3$, $s_1 = \infty$, $s_2 = \infty$

Initial table:

		c_j			2		**0**	Ratio
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	
0	s_1	7	1	1	2	1	0	7/1=1
0	s_2	15	2	**4**	3	0	1	**15/4=3.75**
	Z_j	0	0	0	0	0	0	

	$Z_j - C_j$	0	-3	**-5**	-2	0	0	
	u_j	-	4	3	3	∞	∞	

x_2 enters the basis

$\theta_1 = min\{7,3.75\}$=3.75

$\theta_2 = \infty$

$\theta = u_2 = 3$

$\theta_1 = min\{\theta_1, \theta_2, \theta_3\}$

=min$(3.75, \infty, 3)$=3

Since $\theta = u_r \Rightarrow u_2 = 3$

x_2 enters the basis with substitution

$x_2 = u_2 - x_2' = 3 - x_2'$

$(X_{B1})' = X_{B1} - a_{12}u_2 = 7 - 1.3 = 7 - 3 = 4$

$(X_{B2})' = X_{B2} - a_{22}u_2 = 15 - 12 = 3$

$Z_0' = Z_0 - (Z_2 - C_2)u_2$=0-(-5)3=15

Thus the updated simplex table is $x_2' = -x_2$

		c_j	3	-5	2	0	**0**	Ratio
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	
0	s_1	4	1	-1	2	1	0	4
0	s_2	3	**2**	-4	3	0	1	**3/2=1.5**
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$	0	-3	5	-2	0	0	
	u_j	-	4	3	3	∞	∞	

x_1 enters the basis

$\theta_1 = min\{4,1.5\}$=1.5

$\theta_2 = \infty$

$\theta_3 = u_1 = 4$

$\theta = min\{\theta_1, \theta_2, \theta_3\}$

=min $(1.5, \infty, 4)$=1.5

x_1 enters the basis and also it is a basic variables.

Iteration 1:

		c_j	3	-5	2	0	**0**	Ratio
C_B	Y_B	X_B	x_1	x_2'	x_3	s_1	s_2	
0	s_1	5/2	0	**1**	-1	1	-1	5/2
0	x_1	3/2	1	-2	3/2	0	1/2	**-**
	Z_j	9/2	3	-6	9/2	0	3/2	-

	$Z_j - C_j$		0	-1	5/2	0	3/2	
	u_j		4	3	3	∞	∞	

$\theta_1 = min\{5/2, -, -\}$=5/2
$\theta_2 = min\{(u_i - X_{Bi}/-a_{ir})a\}$
=min $(u_2 - X_{B2}/-a_{22})$=$4 - 3/2| + 2$=5/4
corresponding to x_1.
$\theta_3 = u_r = u_2 = 3$
$\theta = min\{\theta_1, \theta_2, \theta_3\}$
=min(5/2,5/4,3)=5/4
corresponding to x_1.

Since x_1 is non-basic at its upper bound x_1 becomes with the substitution.

$x_1 = u_1 - x_1' = 4 - x_1'$
$(X_{B1})' = X_{B1} - a_{11}u_1 = 5/2 - 0.4 = 5/2$
$(X_{B2})' = X_{B2} - a_{12}u_1 = 3/2 - 1.4 = -5/2$
$Z_0' = Z_0 - (Z_1 - C_1)u_1$=9/2-(3-3)4=9/2

The updated simplex table is $x_1' = -x_1$

		c_j	-3	-5	2	0	**0**	Ratio
C_B	Y_B	X_B	x_1'	x_2'	x_3	s_1	s_2	
0	s_1	5/2	0	1	-1	1	-1	5/2
3	x_1	-5/2	-1	**-2**	3/2	0	1/2	**-5/2/-2=5/4**
	Z_j	-15/2	-3	-6	9/2	0	3/2	-
	$Z_j - C_j$	-15/2	0	-1	5/2	0	3/2	
	u_j	-	4	3	3	∞	∞	

$\theta_1 = min\{5/4, 5/2\}$=5/4
$\theta_2 = \{(u_1 - X_{B2}/-a_{22})\}$=$\frac{4+5/2}{2} = 13/4$
$\theta_3 = u_r = u_2 = 3$
$\theta = min\{\theta_1, \theta_2, \theta_3\}$
=min5/2,13/4,3=5/4
corresponding to x_1

Here x_1 is non-basic at its upper bound (since it is repeated the leaving variable is x_1 and the entering variable is x_2')

Iteration 2:

		c_j	-3	-5	2	0	**0**
C_B	Y_B	X_B	x_1'	x_2'	x_3	s_1	s_2
0	s_1	5/4	0	0			
-5	x_2'	5/4	1/2	1	-3/4	0	-1/4
	Z_j	-25/4	-5/2	-5	15/4	0	5/4
	$Z_j - C_j$		1/2	0	7/4	0	5/4
	u_j		4	3	3	∞	∞

Since all $Z_j - C_j \geq 0$

The optimum solution is obtained

$x_1' = 0,$

$x_1 = u_1 - x_1' \Rightarrow x_1 = 4$

$x_2' = 5/4,$

$x_2 = u_2 - x_2' \Rightarrow x_2 = 7/4$

$x_3 = 0$

max Z=$3x_1 + 5y + 2x_3$

=$3x_1 + 5(x_2 + 7) + 2x_3$

=3(4)+5(7/4+7)+2(0)

= 12+5(7+28/4) =12+5(35/4)

=12+175/4

=223/4

5. Use the bounded variable algorithm to solve max Z= $3x_1 + 5x_2 + 2x_3$

Subject to $x_1 + x_2 + 2x_3 \leq 7$

$2x_1 + 4x_2 + 3x_3 \leq 23$

$0 \leq x_1 \leq 4, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 3$

Solution:

The problem becomes

max Z=$3x_1 + 5x_2 + 2x_3 + 0s_1 + 0s_2$

Subject to $x_1 + x_2 + 2x_3 + s_1 = 7$

$2x_1 + 4x_2 + 3x_3 + s_2 = 23$

$u_1 = 4, u_2 = 5, u_3 = 3, s_1 = \infty, s_2 = \infty$

Initial table:

		c_j	3	5	2	0	**0**	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	Ratio
0	s_1	7	1	1	2	1	0	7/1=7
3	s_2	23	2	**4**	3	0	1	**23/4=5.7**
	Z_j	0	0	0	0	0	0	

	$Z_j - C_j$		-3	-5	-2	0	0	
		u_j	4	5	3	∞	∞	

x_2 enters the basis

$\theta_1 = min\{7,5.7\}$=5.7

$\theta_2 = \infty$,

$\theta_3 = u_r = u_2 = 5$

$\theta = min\{\theta_1, \theta_2, \theta_3\}$

=min$5.7, \infty, 5$=5

5 corresponds to s_2

s_2 leaves the basis

We use the substitution.

$x_2 = u_2 - x_2' = 5 - x_2'$

$(X_{B1})' = X_{B1} - a_{12}u_2 = 7 - 1(5) = 2$

$(X_{B2})' = X_{B2} - a_{22}u_2 = 23 - 4.5 = 3$

$Z_0' = Z_0 - (Z_2 - C_2)u_2 = 0 - (0 - 5).5 = 25$

The updated simplex table is $x_2' = -x_2$

		c_j	3	-5	2	0	**0**	
C_B	Y_B	X_B	x_1	x_2'	x_3	s_1	s_2	Ratio
0	s_1	2	1	-1	2	1	0	2/1=2
3	s_2	3	**2**	-4	3	0	1	**3/2=1.5**
	Z_j	25	0	0	0	0	0	
	$Z_j - C_j$		-3	5	-2	0	0	
		u_j	4	5	3	∞	∞	

$\theta_1 = min\{2,1.5\}$=1.5

$\theta_2 = \infty$,

$\theta_3 = u_r = u_1 = 4$

$\theta = min\{\theta_1, \theta_2, \theta_3\}$

=min$1.5, \infty, 4$=1.5

Iteration 1:

		c_j	3	-5	2			
C_B	Y_B	X_B	x_1	x_2'	x_3	s_1	s_2	Ratio
0	s_1	1/2	0	1	1/2	1	-1/2	1/2
3	x_1	3/2	1	-2	3/2	0	½	**-**

	Z_j	9/2	3	-6	9/2	0	3/2	
	$Z_j - C_j$		0	-1	5/2	0	3/2	
		u_j	4	5	3	∞	∞	

$\theta_1 = min\{1/2, -\}$=1/2

$\theta_2 = \{(u_i - X_{Bi}/-a_{ir})\}$=$u_1 - X_{B2}/-a_{22} = \frac{4-3/2}{2} = 5/4$

$\theta_3 = u_r = u_2 = 5$

$\theta = min\{\theta_1, \theta_2, \theta_3\}$

=min1/2,5/4,5=1/2

$\theta = u_r = u_2 = 5$

$\theta \neq u_r$

Iteration 2:

		c_j	3	-5	2	0	**0**
C_B	Y_B	X_B	x_1	$x_2{'}$	x_3	s_1	s_2
-5	$x_2{'}$	1/2	0	1	1/2	1	-1/2
3	x_1	5/2	1	0	5/2	2	-1/2
	Z_j	10/2=5	3	-5	5	1	1
	$Z_j - C_j$		0	0	3	1	1
		u_j	4	5	3	∞	∞

Since all $Z_j - C_j \geq 0$

The optimum solution is

$x_1 = 5/2, x_2{'} = 1/2$

$x_2 = 5 - x_2{'} = 5 - 1/2$

$x_2 = 19/2$

$x_3 = 0$

max Z=5

6.4 Check your progress

1. Write optimality and feasibility conditions
2. Define basic feasible solution
3.Define upper and lower bounds
4.State bounded variables algorithm
5. Defne basic solution

6.5 Summary

The general LP problem can be written as follows,

maximize (or) minimize Z=$\sum_{j=1}^{n} c_j x_j$

subject to, $\sum_{j=1}^{n} p_j x_j = b$, $x_j \geq 0$, j=1,2,...n

For a given basic vector X_B and its corresponding basis B and objective vector C_B the general simplex tableau developed in previous section shows that any simplex iteration can be represented by the following equations.

Z+$\sum_{j=1}^{n} (Z_j - c_j)x_j$=$C_B B^{-1} b$

$$(X_B)_i + \sum_{j=1}^{n} (B^{-1}p_j)_i x_j = (B^{-1}b)_i$$

$Z_j - C_j$ the reduced cost is defined as $Z_j - C_j = C_B B^{-1} P_j - C_j$

The notation $(v)_i$ is used to represent the i^{th} element of the vector v.

Optimality Condition:

From the Z-equation given above an increase in non-basic x_j above its current zero value will improve the value of Z relative to its current value $(= C_B B^{-1} b)$ only if its $Z_j - C_j$ is strictly negative in the case of maximization and strictly positive in the case of minimization. Otherwise x_j cannot improve the solution and must remain non-basic at zero level.

Though any non-basic variable satisfying the given condition can be choosen to improve the solution the simplex method uses a rule of thumb that calls for selecting the entering variable as the one with the most negative (most positive) $Z_j - C_j$ in case of maximization(minimization).

Feasibility Condition:

The determination of the leaving vector is based on examining the constraint equation associated with the type i^{th} basic variable specifically we have,

$$(X_B)_i + \sum_{j=1}^{n} (B^{-1}p_j)_i x_j = (B^{-1}b)_i$$

when the vector P_j is selected by the optimality condition to enter the basis its associated variable x_j will increase above zero level. At the same time, all the remaining non-basic variables remain at zero level. Thus the i^{th} constraint equation reduces to

$$(X_B)_i = (B^{-1}b)_i - (B^{-1}p_j)_i x_j$$

The equation show that $(B^{-1}p_j)_i > 0$ an increase in x_j can cause $(X_B)_i$ to become negative which violates the non-negatively condition $(X_B)_i \geq 0$ for all i

Thus we have $(B^{-1}b)_i - (B^{-1}p_j)_i x_j \geq 0$ for all i.

The condition yields the maximum value of the entering variable x_j as,

$$x_j = min\{\frac{(B^{-1}b)_i}{(B^{-1}P_j)_i} / (B^{-1}P_j)_i > 0\}$$

The basic variable responsible for producing the minimum ratio leaves the basis solution to become non-basic at zero level.

In certain LPP some or all of the variables are have lower and upper limit to the values such a problem can be written as

max Z=CX subject to

AX=b

$l_j \leq x \leq u_j, (j = 1,2,\dots n)$

where l_i and u_j are lower and upper limit of the variables x_j.

We make convert these constraints to equality by introducing slack and surplus variables as follows.

$$x_j + x_j' = u_j$$

$$x_j - x_j' = l_j$$

$$x_j = l_j + x_j'$$

We use the special technique for upper bound constraints.

If $\theta = \theta_1$, corresponding to X_{Bk}/a_k then X_k is leaves the basis and regular row operation can be performed.

If $\theta = \theta_2$ then the corresponding X_k leaves the basis, x_j becomes non-basic at its upper bound. substituted $x_k = u_k - x_k'$. Introduce x_r and drop x_k and perform row operation.

$(X_{Bi})' = X_{Bi} - a_{ik}u_k$ and

$Z_0' = Z_0 - (Z_k - C_k)u_k$

If $\theta = u_r$ then x_r remains non-basic.

6.6 Keywords

Optimality Condition:

From the Z-equation given above an increase in non-basic x_j above its current zero value will improve the value of Z relative to its current value $(= C_B B^{-1} b)$ only if its $Z_j - C_j$ is strictly negative in the case of maximization and strictly positive in the case of minimization. Otherwise x_j cannot improve the solution and must remain non-basic at zero level.

Though any non-basic variable satisfying the given condition can be choosen to improve the solution the simplex method uses a rule of thumb that calls for selecting the entering variable as the one with the most negative (most positive) $Z_j - C_j$ in case of maximization(minimization).

Feasibility Condition:

The determination of the leaving vector is based on examining the constraint equation associated with the type i^{th} basic variable specifically we have,

$$(X_B)_i + \sum_{j=1}^{n} (B^{-1}p_j)_i x_j = (B^{-1}b)_i$$

when the vector P_j is selected by the optimality condition to enter the basis its associated variable x_j will increase above zero level. At the same time, all the remaining non-basic variables remain at zero level. Thus the i^{th} constraint equation reduces to

$$(X_B)_i = (B^{-1}b)_i - (B^{-1}p_j)_i x_j$$

The equation show that $(B^{-1}p_j)_i > 0$ an increase in x_j can cause $(X_B)_i$ to become negative which violates the non-negatively condition $(X_B)_i \geq 0$ for all i

Thus we have $(B^{-1}b)_i - (B^{-1}p_j)_i x_j \geq 0$ for all i.

The condition yields the maximum value of the entering variable x_j as,

$$x_j = min\{\frac{(B^{-1}b)_i}{(B^{-1}P_j)_i} / (B^{-1}P_j)_i > 0\}$$

The basic variable responsible for producing the minimum ratio leaves the basis solution to become non-basic at zero level.

6.7 Self Assessment Questions and Exercises

1. Prove that, in any simplex iteration, $z_j - c_j = 0$ for all the associated basic variables.
2. Given the general LP in equation form with m equations and n unknowns, determine the maximum number of adjacent extreme points that can be reached from a non degenerate extreme points of the solution space.
3. What are the relationships between extreme points and basic solutions under degeneracy and non degeneratecy? What is the maximum number of iterations that can be performed at a given extreme point assuming no cycling?
4. Solve the following LP by the revised simplex method given the starting basic feasible vector $X_{B0} = (x_2, x_4, x_5)^T$.

   Minimize z = $7x_2 + 11x_3 - 10x_4 + 26x_6$

   Subject to $x_2 - x_3 + x_5 + x_6 = 6$

   $x_2 - x_3 + x_4 + 3x_6 = 8$

   $x_1 + x_2 - 3x_3 + x_4 + x_5 = 12,$

   $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
5. Use the bounded variable algorithm to solve

   Maximize z = $3x_1 + 5x_2 + 2x_3$

   Subject to $x_1 + 2x_2 + 2x_3 \leq 14$

   $2x_1 + 4x_2 + 3x_3 \leq 23$

   $0 \leq x_1 \leq 4, 2 \leq x_2 \leq 5, 0 \leq x_3 \leq 3$
6. Solve the following LPP by bounded variable simplex method.

   Maximize z = $3x_1 + 5x_2 + 3x_3$

   Subject to $x_1 + 2x_2 + 2x_3 \leq 10$

   $2x_1 + 4x_2 + 3x_3 \leq 23$

   $0 \leq x_1 \leq 4, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 3$

6.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

UNIT-VII GAME THEORY- OPTIMAL SOLUTION OF TWO PERSON ZERO SUM GAMES

Structure

7.1 Introduction

This chapter deals with two person zero sum games. The optimality condition of two person zero sum games are explained with suitable examples.Strategies or alternatives, payoff, value of the game and saddle point are described here.

7.2 Objectives

After going through this unit, you will be able to:

- Define payoff
- Define value of the game
- Define saddle- point
- Define two person zero sum game

7.3 Optimal Solution of Two person Zero Sum Games

Game theory deals with the decision situation in which two intelligent opponents conflicting with objectives are trying to ought each other.

Definition:

In a game conflict two opponents known as players will each have a (finite or infinite) number of alternatives or strategies.

Associated with each pair of strategies is a payoff that one player receives from the other such games are known as two person zero sum games.

There are two types of strategies.

i) pure strategy

ii) mixed strategy

Each player knows all strategies out of which he selects the one irrespective of others choice which optimizes his payoff.

i**) Pure strategy** is the decision rule used by a player to select the particular course of action.

ii**) Mixed strategy:**

When both player are left to guess the course with some probabilities, it is called mixed strategy.

Two person zero sum game:

A game with two players A and B in which a gain for A is equal to loss for B. is called two person zero sum game.

The strategies of A are denoted by $A_1, A_2, \dots A_m$ and those of B are $B_1, B_2, \dots B_n$ then the payoff matrix is of the following form

$$\begin{bmatrix} & B_1 & B_2 & \cdots & B_n \\ A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Characteristic of two person zero sum game:

1. There are exactly two players and each player has a finite number of strategy.
2. The given matrix refer to the payoff matrix of A.
3. Each player knows all possible payoff for himself as well as the other player.

Pure strategy game with saddle point:

Maximin and Minimax Principle:

If the maximin and minimax values are equal then the game is said to have saddle point, and the corresponding payoff is called the value of the game.

Note:

1. In general maximin value $\leq$ value of the game $\leq$ minimax value.
2. If the maximin value and minimax value is equal to 0 then the game is said to have fair.
3. If maximin=minimax equal to value of the game then the game is known as determinable (or) stable.

1. Solve the game with the following payoff matrix

$$\begin{pmatrix} & \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{B_3} & \boldsymbol{B_4} \\ \boldsymbol{A_1} & \mathbf{1} & \mathbf{7} & \mathbf{3} & \mathbf{4} \\ \boldsymbol{A_2} & \mathbf{5} & \mathbf{6} & \mathbf{4} & \mathbf{5} \\ \boldsymbol{A_3} & \mathbf{7} & \mathbf{2} & \mathbf{0} & \mathbf{3} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} & B_1 & B_2 & B_3 & B_4 & Rowmin \\ A_1 & 1 & 7 & 3 & 4 & 1 \\ A_2 & 5 & 6 & 4 & 5 & 4 \\ A_3 & 7 & 2 & 0 & 3 & 0 \\ Columnmax & 7 & 7 & 4 & 5 & \end{pmatrix}$$

maximin=4,minimax=4

value of the game=4

Here the maximin=minimax=4

The saddle point is (2,3)

The optimal strategy of A is A_2 and that of B is B_3.

2. $\begin{pmatrix} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 20 & 15 & 12 & 35 \\ A_2 & 25 & 14 & 8 & 10 \\ A_3 & 40 & 2 & 10 & 5 \\ A_4 & -5 & 4 & 11 & 0 \end{pmatrix}$

Solution:

$$\begin{pmatrix} & B_1 & B_2 & B_3 & B_4 & Rowmin \\ A_1 & 20 & 15 & 12 & 35 & 12 \\ A_2 & 25 & 14 & 8 & 10 & 8 \\ A_3 & 40 & 2 & 10 & 5 & 2 \\ A_4 & -5 & 4 & 11 & 0 & -5 \\ Columnmax & 40 & 15 & 12 & 35 & \end{pmatrix}$$

minimax=12, maximin=12

Here maximin=minimax=12

Value of the game=12

The saddle point is (1,3)

The optimal strategy of A is A_1 and that of B is B_3

Hence the game is determinable.

3. If the following game is determinable. Find the limits for the value of X.

$\begin{pmatrix} \lambda & 6 & 4 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{pmatrix}$ **(or)** $\begin{pmatrix} \lambda & -1 & 2 \\ 6 & \lambda & -7 \\ 4 & -7 & \lambda \end{pmatrix}$

Solution:

Ignoring the value of λ.

The maximin and minimax value are as follows.

$$\begin{pmatrix} & & & & Rowmin \\ & \lambda & 6 & 4 & 4 \\ & -1 & \lambda & -7 & -7 \\ & -2 & 4 & \lambda & -2 \\ Columnmax & -1 & 6 & 4 & \end{pmatrix}$$

maximin=4,minimax=-1

Since the game is determinable

minimax=maximin=value of the game

λ lies between 4 and -1.

4. If the saddle point is (2,2) in the following payoff matrix, Find the values of p and q $\begin{pmatrix} 2 & 6 & 5 \\ 10 & 7 & q \\ 5 & p & 8 \end{pmatrix}$

Solution:

Ignoring the value of p and q the maximin and minimax values are as follows.

NOTES

$$\begin{pmatrix} & & & & Rowmin \\ & 2 & 6 & 5 & 2 \\ & 10 & 7 & q & 7 \\ & 5 & p & 8 & 5 \\ columnmax & 10 & 7 & 8 & \end{pmatrix}$$

minimax=7,maximin=7

Given the saddle point is (2,2)

The value of the game=7 and $p \leq 7, q \leq 7$

7.4 Check your progress

- Define payoff
- Define value of the game
- Define saddle- point
- Define two person zero sum game

7.5 Summary

- In a game conflict two opponents known as players will each have a (finite or infinite) number of alternatives or strategies.
- **Pure strategy** is the decision rule used by a player to select the particular course of action.
- **Mixed strategy:**
- When both player are left to guess the course with some probabilities, it is called mixed strategy.

Two person zero sum game:

- A game with two players A and B in which a gain for A is equal to loss for B. is called two person zero sum game.
- If the maximin and minimax values are equal then the game is said to have saddle point, and the corresponding payoff is called the value of the game.
- **Characteristic of two person zero sum game:**
- 1. There are exactly two players and each player has a finite number of strategy.
- 2. The given matrix refer to the payoff matrix of A.
- 3. Each player knows all possible payoff for himself as well as the other player.

7.6 Keywords

Pure strategy: It is the decision rule used by a player to select the particular course of action.

Mixed strategy: When both player are left to guess the course with some probabilities, it is called mixed strategy.

Two person zero sum game:
A game with two players A and B in which a gain for A is equal to loss for B. is called two person zero sum game

Saddle Point and Payoff: If the maximin and minimax values are equal then the game is said to have saddle point, and the corresponding payoff is called the value of the game

7.7 Self Assessment Questions and Exercises

1. 1Determine the saddle point solution, the associated pure strategies and the value of the game for each of the following game. The pay-off for player A.

Player B

		B_1	B_2	B_3	B_4	B_5
	B_1	-2	0	0	5	3
$Player\ A$	B_2	3	2	1	2	2
	B_3	-4	-3	0	-2	6
	B_4	5	3	-4	2	-6

2. The following games give A's pay-off. Determine the values of p and q that will make the entry (2,2) of each game a saddle point.

Player B

		B_1	B_2	B_3
	A_1	1	q	6
$Player\ A$	A_2	p	5	10
	A_3	6	2	3

3. Solve each of the following game whose pay-off matrix is given below.

Player B

		B_1	B_2	B_3
	A_1	9	3	1
$Player\ A$	A_2	6	5	4
	A_3	2	4	3

4. Solve the following game whose pay-off matrix is given by:

Player B

		B_1	B_2	B_3	B_4	B_5
	B_1	9	3	1	8	0
$Player\ A$	B_2	6	5	4	6	7
	B_3	2	4	3	3	8
	B_4	5	6	2	2	1

5. Check whether the following two-person zero-sum game is strictly determinable.

Player B

5	0
0	2

Player A

7.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

NOTES

UNIT- VIII SOLUTION OF MIXED STRATEGY GAMES

Structure

8.1 Introduction

So far, we were concerned with a two person zero sum games,this chapter introduces the concept of mixed strategy games.The dominance property of games is also introduced with suitable examples.

8.2 Objectives

After going through this unit, you will be able to:

- Define a game without a saddle point
- Under the concept of mixed strategy games
- Define dominance property of a game

8.3 Curves on surfaces

Mixed strategy games without saddle point:

Formula for finding the values of the game incase 2×2 games without saddle point.

Let the payoff matrix of A be

$$\begin{pmatrix} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{pmatrix}$$

The strategies and the probabilities of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}; S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

and probability $p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$$p_2 = 1 - p_1, q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$q_2 = 1 - q_1$$

Then the value of the game

$$v = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

5. Solve the following game $\begin{pmatrix} & \boldsymbol{B_1} & \boldsymbol{B_2} \\ \boldsymbol{A_1} & \mathbf{8} & \mathbf{-3} \\ \boldsymbol{A_2} & \mathbf{-3} & \mathbf{1} \end{pmatrix}$

Solution:

$a_{11} = 8, a_{12} = -3, a_{21} = -3, a_{22} = 1$

$p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$ [There is no saddle point we use the mixed strategy method]

The optimal strategies of A and B are

$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

$p_1 = \frac{1+3}{(8+1)-(-3-3)} = 4/9 + 6 = 4/15$

$p_2 = 1 - p_1 = 1 - 4/15 = 11/15$

$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$= \frac{1+3}{15} = 4/15$

$q_2 = 1 - q_1 = 1 - 4/15 = 11/15$

$v = \frac{a_{11}.a_{22}-a_{12}.a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$= \frac{8.1-(-3.-3)}{15} = 8 - 9/15 = -1/15$

6. Solve the following game $\begin{pmatrix} \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} \end{pmatrix}$

Solution:

$$\begin{pmatrix} & B_1 & B_2 \\ A_1 & 5 & 1 \\ A_2 & 3 & 4 \end{pmatrix}$$

There is no saddle point.

Hence we use the mixed strategy method.

$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$ $a_{11} = 5, a_{12} = 1, a_{21} = 3, a_{22} = 4$

$p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$= \frac{4-3}{(5+4)-(1+3)} = 1/9 - 4 = 1/5$

$p_2 = 1 - p_1 = 1 - 1/5 = 4/5$

$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$= \frac{4-1}{5} = 3/5$

$q_2 = 1 - q_1 = 1 - 3/5 = 2/5$

$v = \frac{a_{11}.a_{22}-a_{12}.a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$= \frac{5.4-1.3}{5} = 20 - 3/5 = 17/5$

The optimal strategies are $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/5 & 4/5 \end{pmatrix}$ $S_B = \begin{pmatrix} B_1 & B_2 \\ 3/5 & 2/5 \end{pmatrix}$

Graphical Method:

The Graphical method can be applied to solve the game whose payoff matrix is of the $2 \times n$ or $m \times 2$.

Using the graphical method we can reduce the given payoff matrix into 2×2 type which can be solved by mixed strategy.

Procedure of $2 \times n$ matrix:

The $2 \times n$ payoff matrix can be defined as follows

$$\begin{pmatrix} & B_1 & B_2 & \cdots & B_n & probability \\ A_1 & a_{11} & a_{12} & \cdots & a_{1n} & p_1 \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} & p_2 \\ probability & q_1 & q_2 & \cdots & q_n & \end{pmatrix}$$

This format is called $2 \times n$ matrix format for the game.
Now consider the two vertical axis corresponding to $p_1 = 0 and p_1 = 1$.
These axes are parallel and linear at a distances one unit apart from joining the two points. Then we can get the corresponding linear equations to determine the lower envelope of all these lines.

The highest point of this envelope represents the maximum among the minimum and the expected payoff is linear which intersect at these points gives the optimal strategy

B's Strategy **A' Expected Payoff**

B_1 $E_1(p_1) = a_{11}p_1 + a_{21}p_2$

$=(a_{11} - a_{21})p_1 + a_{21}$

B_2 $E_2(p_1) = a_{12}p_1 + a_{22}p_2$

$=(a_{12} - a_{22})p_1 + a_{22}$

$\vdots$

B_n $E_n(p_1) = a_{1n}p_1 + a_{2n}p_2$

$=(a_{1n} - a_{2n})p_1 + a_{2n}$

Procedure of $m \times 2$ matrix:

The $m \times 2$ payoff matrix be

$$\begin{pmatrix} & B_1 & B_2 & probability \\ A_1 & a_{11} & a_{12} & p_1 \\ A_2 & a_{21} & a_{22} & p_2 \\ \vdots & \vdots & \vdots & \vdots \\ A_m & a_{m1} & a_{m2} & p_m \\ Probability & q_1 & q_2 & \end{pmatrix}$$

A's Strategy **B's Expected Payoff**

A_1 $E_1(q_1) = a_{11}q_1 + a_{12}q_2$

$=(a_{11} - a_{12})q_1 + a_{12}$

A_2 $E_2(q_2) = a_{21}q_1 + a_{22}q_2$

$=(a_{21} - a_{22})q_1 + a_{22}$

$\vdots$

A_m $E_m(q_1) = a_{m1}q_1 + a_{m2}q_2$

$=(a_{m1} - a_{m2})q_1 + a_{m2}$

6. Solve the following game graphically

$$\begin{pmatrix} & \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{B_3} & \boldsymbol{B_4} \\ \boldsymbol{A_1} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{-2} \\ \boldsymbol{A_2} & \mathbf{1} & \mathbf{0} & \mathbf{3} & \mathbf{2} \end{pmatrix}$$

Solution:

Let $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$

Then we have the expected payoff as follows.

B's Strategy	**A's Expected Payoff**
B_1	$E_1(p_1) = (2-1)p_1 + 1 = p_1 + 1$
B_2	$E_2(p_1) = (1-0)p_1 + 0 = p_1$
B_3	$E_3(p_1) = (0-3)p_1 + 3 = -3p_1 + 3$
B_4	$E_4(p_1) = (-2-2)p_1 + 2 = -4p_1 + 2$

When $p_1 = 0$

$E_1(0) = 0 + 1 = 1$

$E_2(0) = 0$

$E_3(0) = -3(0) + 3 = 3$

$E_4(0) = -4(0) + 2 = 2$

When $p_1 = 1$

$E_1(1) = 2$

$E_2(1) = 1$

$E_3(1) = 0$

$E_4(1) = -2$

From this point we observe that the highest point of the lower envelope is H, which is the intersection of $E_2(p_1)$ and $E_4(p_1)$

B_2 and B_4 are the optimal strategy of B.

we have the following payoff matrix

$$\begin{pmatrix} & B_2 & B_4 \\ A_1 & 1 & -2 \\ A_2 & 0 & 2 \end{pmatrix}$$

$a_{11} = 1, a_{12} = -2, a_{21} = 0, a_{22} = 2$

There is no saddle point.

Hence we use the mixed strategy method.

The optimal strategy of A and B are

$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ $S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}$

$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$= \frac{2-0}{(1+2)-(2-0)} = 2/3 + 2 = 2/5$

$p_2 = 1 - p_1$

=1-2/5=3/5

$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$= \frac{2+2}{5} = 4/5$

$q_2 = 1 - q_1 = 1 - 4/5 = 1/5$

$v = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

NOTES

$=\frac{1.2-(-2.0)}{5} = 2/5$

The optimal strategies are

$S_A = \begin{pmatrix} A_1 & A_2 \\ 2/5 & 3/2 \end{pmatrix} S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{pmatrix}$

7. Solve the following game

$$\begin{pmatrix} & \boldsymbol{B_1} & \boldsymbol{B_2} \\ \boldsymbol{A_1} & \boldsymbol{-6} & \boldsymbol{7} \\ \boldsymbol{A_2} & \boldsymbol{4} & \boldsymbol{-5} \\ \boldsymbol{A_3} & \boldsymbol{-1} & \boldsymbol{-2} \\ \boldsymbol{A_4} & \boldsymbol{-2} & \boldsymbol{5} \\ \boldsymbol{A_5} & \boldsymbol{7} & \boldsymbol{-6} \end{pmatrix}$$

Solution:

Let $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix} S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

Then we have the expected payoff as follows

A's Strategy	**B's Expected Payoff**
A_1	$E_1(q_1) = (-6-7)q_1 + 7 = -13q_1 + 7$
A_2	$E_2(q_1) = (4+5)q_1 - 5 = 9q_1 - 5$
A_3	$E_3(q_1) = (-1+2)q_1 - 2 = q_1 - 2$
A_4	$E_4(q_1) = (-2-5)q_1 + 5 = -7q_1 + 5$
A_5	$E_5(q_1) = (7+6)q_1 - 6 = 13q_1 - 6$

when $q_1 = 0$	when $q_1 = 1$
$E_1(0) = 7$	$E_1(1) = -6$
$E_2(0) = -5$	$E_2(1) = 4$
$E_3(0) = -2$	$E_3(1) = -1$
$E_4(0) = +5$	$E_4(1) = -2$
$E_5(0) = -6$	$E_5(1) = 7$

From this point we observe that the lowest point of the higher envelope is H, which is the intersection of $E_4(q_1)$ and $E_5(q_1)$

A_4 and A_5 are the optimal strategy of A.

we have the following payoff matrix.

$$\begin{pmatrix} & B_1 & B_2 \\ A_4 & -2 & 5 \\ A_5 & 7 & -6 \end{pmatrix}$$

$a_{11} = -2, a_{12} = 5, a_{21} = 7, a_{22} = -6$

There is no saddle point.

Hence we use the mixed strategy method.

The optimal strategy of A and B are

$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix} \quad S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

$p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$=\frac{-6-7}{(-2-6)-(5+7)} = 13/20$

$p_2 = 1 - p_1 = 1 - 13/20 = 7/20$

$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$=\frac{-6-5}{-20} = -11/-20$

$q_1 = 11/20$

$q_2 = 1 - q_1 = 1 - 11/20 = 9/20$

$v = \frac{a_{11}.a_{22}-a_{12}.a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$=\frac{(-2.-6)-5.7}{-20} = -23/-20 = 23/20$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & 13/20 & 7/20 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ 11/20 & 9/20 \end{pmatrix}$$

8. $\begin{pmatrix} & \boldsymbol{B_1} & \boldsymbol{B_2} \\ \boldsymbol{A_1} & \mathbf{2} & \mathbf{3} \\ \boldsymbol{A_2} & \mathbf{6} & \mathbf{7} \\ \boldsymbol{A_3} & \mathbf{-6} & \mathbf{10} \\ \boldsymbol{A_4} & \mathbf{-3} & \mathbf{-2} \\ \boldsymbol{A_5} & \mathbf{3} & \mathbf{2} \end{pmatrix}$

Solution:

Let $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix}$ $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

Then we have the expected payoff as follows

A's Strategy	B's Expected Payoff
A_1	$E_1(q_1) = (2-3)q_1 + 3 = -q_1 + 3$
A_2	$E_2(q_1) = (6-7)q_1 + 7 = -q_1 + 7$
A_3	$E_3(q_1) = (-6-10)q_1 + 10 = -16q_1 + 10$
A_4	$E_4(q_1) = (-3+2)q_1 - 2 = -q_1 - 2$
A_5	$E_5(q_1) = (3-2)q_1 + 2 = q_1 + 2$

when $q_1 = 0$ when $q_2 = 0$

$E_1(0) = 3$ $E_1(1) = 2$

$E_2(0) = 7$ $E_2(1) = 6$

$E_3(0) = 10$ $E_3(1) = -6$

$E_4(0) = -2$ $E_4(1) = -3$

$E_5(0) = 2$ $E_5(1) = 3$

From this point we observe that the lowest point of the highest envelope is H, which is the intersection of $E_2(q_1)$ and $E_3(q_1)$

A_2 and A_3 are the optimal strategy of A.

$$\begin{pmatrix} & B_1 & B_2 \\ A_2 & 6 & 7 \\ A_3 & -6 & 10 \end{pmatrix}$$

$a_{11} = 6, a_{12} = 7, a_{21} = -6, a_{22} = 10$

$p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$=\frac{10+6}{(6+10)-(7-6)} = 16/15$

$p_2 = 1 - p_1 = 1 - 16/15 = -1/15$

$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$=\frac{10-7}{15} = 3/15$

$q_1 = 3/15$

$q_2 = 1 - q_1 = 1 - 3/15 = 12/15$

$v = \frac{a_{11}.a_{22}-a_{12}.a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$=\frac{(6.10)-7.(-6)}{15} = 102/15$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 16/15 & -1/5 & 0 & 0 \end{pmatrix} \quad S_B = \begin{pmatrix} B_1 & B_2 \\ 3/15 & 12/15 \end{pmatrix}$$

Dominance Property:

Dominance property is used to reduce the size of payoff matrix.

Rules:

1. In the payoff matrix if each element of the i^{th} row is less than or equal to the corresponding element of the j^{th} row, then the i^{th} row is deleted.

Because j^{th} row dominate the i^{th} row

2: If every element of r^{th} column $\geq$ a corresponding element of the s^{th} column, then the r^{th} column can be deleted.

3: If some linear combination of some rows dominates the i^{th} row then the i^{th} row can be deleted.

9. Solve the following game

$$\begin{pmatrix} & \boldsymbol{B_1} & \boldsymbol{B_2} & \boldsymbol{B_3} \\ \boldsymbol{A_1} & \mathbf{1} & \mathbf{7} & \mathbf{2} \\ \boldsymbol{A_2} & \mathbf{0} & \mathbf{2} & \mathbf{7} \\ \boldsymbol{A_3} & \mathbf{5} & \mathbf{1} & \mathbf{6} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} & & & & rowmin \\ & 1 & 7 & 2 & 1 \\ & 0 & 2 & 7 & 0 \\ & 5 & 1 & 6 & 1 \\ columnmax & 5 & 7 & 7 & \end{pmatrix}$$

There is no saddle point column 3 dominates column 1.

Delete column 3

The reduced matrix is $\begin{pmatrix} & B_1 & B_2 \\ A_1 & 1 & 7 \\ A_2 & 0 & 2 \\ A_3 & 5 & 1 \end{pmatrix}$

Row 1 dominates row 2. Delete Row 2

The reduced matrix is $\begin{pmatrix} & B_1 & B_2 \\ A_1 & 1 & 7 \\ A_3 & 5 & 1 \end{pmatrix}$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \qquad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

$$a_{11} = 1, a_{12} = 7, a_{21} = 5, a_{22} = 1$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1-5}{(1+1)-(7+5)} = 4/10 = 2/5$$

$$p_2 = 1 - p_1 = 1 - 2/5 = 3/5$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{1-7}{-10} = 3/5$$

$$q_1 = 3/5$$

$$q_2 = 1 - q_1 = 1 - 3/5 = 2/5$$

$$v = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1.1) - 7.5}{-10} = -34/-10$$

$$v = 17/5$$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/5 & 0 & 3/5 \end{pmatrix} \qquad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 3/5 & 2/5 & 0 \end{pmatrix}$$

10. Solve the following game

$$\begin{pmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} & & & & & Rowmin \\ & -5 & 3 & 1 & 20 & -5 \\ & 5 & 5 & 4 & 6 & 4 \\ & -4 & -2 & 0 & -5 & -5 \\ columnmax & 5 & 5 & 4 & 20 & \end{pmatrix}$$

Value of the game=4

Saddle point is (2,3)

The optimal strategy A is A_2 and that B is B_3.

11. Using dominance to simplify and solve the following game graphically

$$\begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 18 & 4 & 6 & 4 \\ 2 & 16 & 2 & 13 & 7 \\ 3 & 11 & 5 & 17 & 3 \\ 4 & 7 & 6 & 12 & 2 \end{pmatrix}$$

NOTES

Solution:

$$\begin{pmatrix} & & & & & Rowmin \\ & 18 & 4 & 6 & 4 & 4 \\ & 16 & 2 & 13 & 7 & 2 \\ & 11 & 5 & 17 & 3 & 3 \\ & 7 & 6 & 12 & 2 & 2 \\ Columnmax & 18 & 6 & 17 & 7 & \end{pmatrix}$$

There is no saddle point.
Column 2 dominate column 1
delete column 1.

The reduce matrix is $\begin{pmatrix} & 2 & 3 & 4 \\ 1 & 4 & 6 & 4 \\ 2 & 2 & 13 & 7 \\ 3 & 5 & 17 & 3 \\ 4 & 6 & 12 & 2 \end{pmatrix}$

Column 3 dominate column 2 delete column 3.
The reduced matrix is

$$\begin{pmatrix} & 2 & 4 & Rowmin \\ 1 & 4 & 4 & 4 \\ 2 & 2 & 7 & 2 \\ 3 & 5 & 3 & 3 \\ 4 & 6 & 2 & 2 \\ Columnmax & 6 & 7 & \end{pmatrix}$$

A’s strategy B’s expected payoff
A_1 $E_1(q_1) = (4-4)q_1 + 4 = 4$
A_2 $E_2(q_1) = (2-7)q_1 + 7 = -5q_1 + 7$
A_3 $E_3(q_1) = (5-3)q_1 + 3 = 2q_1 + 3$
A_4 $E_4(q_1) = (6-2)q_1 + 2 = 4q_1 + 2$
when $q_1 = 0$ when $q_1 = 1$
$E_1(q_1) = 4$ $E_1(q_1) = 4$
$E_2(q_1) = 7$ $E_2(q_1) = 2$
$E_3(q_1) = 3$ $E_3(q_1) = 5$
$E_4(q_1) = 2$ $E_4(q_1) = 6$

The point of intersection H, intersection is (2), (4).

Hence the matrix is $\begin{pmatrix} & 2 & 4 \\ 2 & 2 & 7 \\ 4 & 6 & 2 \end{pmatrix}$

There is no saddle point.

$$P_1 = -4/4 - 13 = 4/9$$
$$P_2 = 1 - 4/9 = 5/9$$
$$q_1 = 5/9$$
$$q_2 - 4/9$$
$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 4/9 & 0 & 5/9 \end{pmatrix}$$
$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 5/9 & 0 & 4/9 \end{pmatrix}$$

12. Solve $\begin{pmatrix} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 3 & 2 & 4 & 0 \\ A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 8 \end{pmatrix}$

Solution:

There is no saddle point.
Row 3 dominate row 1 delete row 1

The reduced matrix is $\begin{pmatrix} & B_1 & B_2 & B_3 & B_4 \\ A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 8 \end{pmatrix}$

Column 1 dominate column 3
delete column 1

The reduced matrix is $\begin{pmatrix} & B_2 & B_3 & B_4 & \\ A_2 & 4 & 2 & 4 & 2 \\ A_3 & 2 & 4 & 0 & 0 \\ A_4 & 4 & 0 & 8 & 0 \\ & 4 & 4 & 8 & \end{pmatrix}$

There is no single row and no single column dominated another row and column.
Hence column 2 dominate the average of column 3 and 4
The average of column 3 and 4 is 2+4/2, 4+0/2, 0+8/2
3,2,4
Column 2 dominate the average of column 3 and 4 delete column 2

$$\begin{pmatrix} & B_3 & B_4 & \\ A_2 & 2 & 4 & 2 \\ A_3 & 4 & 0 & 0 \\ A_4 & 0 & 8 & 0 \\ & 4 & 8 & \end{pmatrix}$$

Hence row 2 dominate the average of row 3 and 4
The average of row 3 and 4 is 4+0/2, 0+8/2
Row 2 dominate the average of row 3 and 4
Delete row 2.

The reduce matrix is $\begin{pmatrix} & B_3 & B_4 \\ A_3 & 4 & 0 \\ A_4 & 0 & 8 \end{pmatrix}$

There is no saddle point.

$$P_1 = 8/12 = 2/3$$
$$P_2 = 1/3$$
$$q_1 = 2/3$$
$$q_2 = 1/3$$

NOTES

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix}.$$

8.4Check your progress

- Define a game without a saddle point
- Under the concept of mixed strategy games
- Define dominance property of a game

8.5 Summary

Let the payoff matrix of A be

$$\begin{pmatrix} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{pmatrix}$$

The strategies and the probabilities of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}; S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

and probability $p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$$p_2 = 1 - p_1, q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$q_2 = 1 - q_1$$

Then the value of the game

$$v = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Dominance Property:

Dominance property is used to reduce the size of payoff matrix.

Rules:

1. In the payoff matrix if each element of the i^{th} row is less than or equal to the corresponding element of the j^{th} row, then the i^{th} row is deleted. Because j^{th} row dominate the i^{th} row

2: If every element of r^{th} column $\geq$ a corresponding element of the s^{th} column, then the r^{th} column can be deleted.

3: If some linear combination of some rows dominates the i^{th} row then the i^{th} row can be deleted.

8.6 Keywords

Let the payoff matrix of A be

$$\begin{pmatrix} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{pmatrix}$$

The strategies and the probabilities of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}; S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

and probability $p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$$p_2 = 1 - p_1, q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$q_2 = 1 - q_1$$

Then the value of the game

$$v = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

8.7 Self Assessment Questions and Exercises

1. Consider the following 2×4 game. The pay-off is for player A.

	B_1	B_2	B_3	B_4
A_1	2	2	3	−1
A_2	4	3	2	6

2. If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. Similar argument follows for columns.

   Use the notion of dominance to simplify the rectangular game with the following pay-off and solve it graphically.

   Player B

   Player A $\begin{pmatrix} 18 & 4 & 6 & 4 \\ 6 & 2 & 13 & 7 \\ 11 & 5 & 17 & 3 \\ 7 & 6 & 12 & 2 \end{pmatrix}$

3. Solve the following game graphically. The pay-off is for player A.

	B_1	B_2
A_1	5	8
A_2	6	5
A_3	5	7

8.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008

BLOCK III: SOLUTION OF GAMES USING LPP AND OPTIMIZATION THEORY

UNIT IX LINEAR PROGRAMMING SOLUTION OF GAMES

Structure

9.1 Introduction

Game thoery bears a strong relationship to linear programming in the sense that a two-person zero-sum game can be expressed as a linear program, and vice versa.This chapter illustrates the solution of games by linear programming.

9.2 Objectives

After going through this unit, you will be able to:

- Define a linear programming solution of games
- Understand the concept of linear programming solution of games

9.3 Linear programming solution of games

Problems which are solved using the algebraic method can be solved using linear programming method also.

For the given $m \times n$ payoff matrix we assume

$$S_A = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$$

and $S_B = \begin{pmatrix} B_1 & B_2 & \dots & B_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$

Using maximin criterion for A, we get,

$$a_{11}p_1 + a_{12}p_2 + \dots + a_{m1}p_m \geq V$$
$$a_{21}p_1 + a_{22}p_2 + \dots + a_{m2}p_m \geq V$$

.
.
.

$$a_{1n}p_1 + a_{2n}p_2 + \dots + a_{mn}p_m \geq V$$

Where $p_1 + p_2 + \dots + p_m = 1, p_i \geq 0$

Dividing both side by V, we get

$$a_{11}\frac{p_1}{V} + a_{12}\frac{p_2}{V} + \dots + a_{m1}\frac{p_m}{V} \geq 1$$
$$a_{21}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \dots + a_{m2}\frac{p_m}{V} \geq 1$$

NOTES

.
.
.

$$a_{1n}\frac{p_1}{V}+a_{2n}\frac{p_2}{V}+\ldots+a_{mn}\frac{p_m}{V}\geq 1$$
$$\frac{p_1}{V}+\frac{p_2}{V}+\ldots+\frac{p_m}{V}=\frac{1}{V}$$

Denote $\frac{p_1}{V},\frac{p_2}{V},\ldots,\frac{p_m}{V}$ by
$x_1,x_2,\ldots x_m$ we get the LPP

$$max\frac{1}{V}=Z=x_1+x_2+\ldots+x_m$$
$$(maxV\Leftrightarrow min\frac{1}{V})$$

Subject to,

$$a_{11}x_1+a_{21}x_2+\ldots+a_{m1}x_m\geq V$$
$$a_{21}x_1+a_{22}x_2+\ldots+a_{m2}x_m\geq V$$

.
.
.

$a_{1n}x_1+a_{2n}x_2+\ldots+a_{mn}c_m\geq V$ equation (1)

$$x_1,x_2,\ldots x_m\geq 0$$

Similarly, considering minimax criterion for A, we get

$$a_{11}\frac{q_1}{V}+a_{12}\frac{q_2}{V}+\ldots+a_{1n}\frac{q_n}{V}\leq 1$$
$$a_{21}\frac{q_1}{V}+a_{22}\frac{q_2}{V}+\ldots+a_{2n}\frac{q_n}{V}\leq 1$$

.
.
.

$$a_{m1}\frac{q_1}{V}+a_{m2}\frac{q_2}{V}+\ldots+a_{mn}\frac{q_n}{V}\leq 1$$
$$\frac{q_1}{V}+\frac{q_2}{V}+\ldots+\frac{q_n}{V}=\frac{1}{V}$$

Denote $\frac{q_1}{V},\frac{q_2}{V},\ldots,\frac{q_n}{V}$ by $y_1,y_2,\ldots y_n$
we get the LPP

$$max\frac{1}{V}=W=y_1+y_2+\ldots+y_n$$
$$(minV\Leftrightarrow max\frac{1}{V})$$

Subject to,

$$a_{11}y_1+a_{12}y_2+\ldots+a_{1n}y_n\leq V$$
$$a_{21}y_1+a_{22}y_2+\ldots+a_{2n}y_n\leq V$$

.
.
.

$a_{m1}y_1+a_{m2}y_2+\ldots+a_{mn}y_n\geq V$ equation (1)

$$y_1,y_2,\ldots y_n\leq 0$$

Find that 1 and 2 are dual problems.

We can solve one of these and obtain the solution of the other using the principle of duality, the values of $x_1,x_2,\ldots x_m$ gives A's strategies and

$y_1, y_2, \ldots y_n$ yields B's strategies and the values of the game can be determined.

13. Solve the game using LPP method

$$\begin{pmatrix} & B_1 & B_2 & B_3 \\ A_1 & 9 & 1 & 4 \\ A_2 & 0 & 6 & 3 \\ A_3 & 5 & 2 & 8 \end{pmatrix}$$

Solution:

The payoff matrix is $\begin{pmatrix} & B_1 & B_2 & B_3 & \\ A_1 & 9 & 1 & 4 & 1 \\ A_2 & 0 & 6 & 3 & 0 \\ A_3 & 5 & 2 & 8 & 2 \\ & 9 & 6 & 8 & \end{pmatrix}$

The matrix has no saddle point.
The strategies of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$
$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

Considering the payoff for A,
The LPP is minimize Z=$\frac{1}{V} = x_1 + x_2 + x_3$
Subject to the constraints

$$9x_1 + 0x_2 + 5x_3 \geq 1$$
$$x_1 + 6x_2 + 2x_3 \geq 1$$
$$4x_1 + 3x_2 + 8x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

equation(1) The corresponding LPP for B's payoff matrix is
maximize W=$V = y_1 + y_2 + y_3$
Subject to the constraints

$$9y_1 + y_2 + 4y_3 \leq 1$$
$$0y_1 + 6y_2 + 3y_3 \leq 1$$
$$5y_1 + 2y_2 + 8y_3 \leq 1$$
$$y_1, y_2, y_3 \geq 0$$

equation (2)
eqn(2) is the dual of eqn(1)
We solve eqn(1)
Now the standard form of the LPP is maximize W=$V = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$
Subject to,

$$9y_1 + y_2 + 4y_3 + s_1 = 1$$
$$0y_1 + 6y_2 + 3y_3 + s_2 = 1$$
$$5y_1 + 2y_2 + 8y_3 + s_3 = 1$$
$$y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

NOTES

Initial table:

		c_j	1	1	1	0	0	0	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3	**Ratio**
0	s_1	1	9	1	4	1	0	0	1/4
0	s_2	1	0	6	3	0	1	0	1/3
0	s_3	1	5	2	8	0	0	1	1/8
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		-1	-1	-1	0	0	0	

Entering variable y_3
Leaving variable s_3
Pivot element=8

Iteration 1:

		c_j	1	1	1	0	0	0	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3	**Ratio**
0	s_1	1/2	13/2				0	-4/3	-
0	s_2	5/8	-15/8	21/4			1	-3/8	0.1
1	y_3	1/8	5/8	2/8	1	0	0	1/8	0.5
	Z_j	1/8	5/8	2/8	1	0	0	1/8	
	$Z_j - C_j$		-3/8	-3/4	0	0	0	1/8	

Entering variable is y_2
Leaving variable is s_2
Pivot element=21/4

Iteration 2:

		c_j	1	1	1	0	0	0	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3	**Ratio**
0	s_1	1/2	13/2	0	0	1	0	-1/2	1/13
1	y_2	5/42	-15/42	1	0	0	4/21	-3/42	-
1	y_3	2/21	15/21	0	1	0	-1/21	3/21	2/15
	Z_j	9/42	15/42	1	1	0	3/21	6/42	
	$Z_j - C_j$		9/42	-27/42	0	0	0	3/21	

y_1 enters the basis,
s_1 leaves the basis.

Iteration 3:

		c_j	1	1	1	0	0	0
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3
1	y_1	1/13	1	0	0	2/13	0	-1/13
1	y_2	40/273	0	1	0	5/91	4/21	-9/91
1	y_3	11/273	0	0	1	-10/91	-1/21	18/91

NOTES

	$Z_j - C_j$		**0**	**0**	**0**	**9/91**	**1/7**	**2/91**

Since all $Z_j - C_j \geq 0$
The optimal solution is obtained
We have $y_1 = 9/91, y_2 = 40/273, y_3 = 11/273$

$$x_1 = 9/91, x_2 = 1/7, x_3 = 2/91$$

$$maxW = \frac{1}{V} = 1/3 + 40/273 + 11/273 = 72/273$$

$$\frac{1}{V} = 24/91$$

$$V = 91/24$$

$$y_1 = q_1/V \Rightarrow q_1 = y_1.V = 7/24$$

$$q_2 = y_2.V = 5/9$$

$$q_3 = y_3.V = 11/72$$

$$x_1 = p_1/V \Rightarrow p_1 = x_1.V = 3/8$$

$$p_2 = x_2.V = 13/24$$

$$p_3 = x_3.V = 1/12$$

The strategies of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 3/8 & 13/24 & 1/12 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 7/24 & 5/9 & 11/72 \end{pmatrix}$$

14. Solve by LPP method

$$\begin{pmatrix} & B_1 & B_2 & B_3 \\ A_1 & 3 & -4 & 2 \\ A_2 & 1 & -3 & -7 \\ A_3 & -2 & 4 & 7 \end{pmatrix}$$

Solution:

The payoff matrix is $\begin{pmatrix} & B_1 & B_2 & B_3 & \\ A_1 & 3 & -4 & 2 & -4 \\ A_2 & 1 & -3 & -7 & -7 \\ A_3 & -2 & 4 & 7 & -2 \\ & 3 & 4 & 7 & \end{pmatrix}$

The matrix has no saddle point.
The strategies of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

Considering the payoff for A, the LPP is minimize Z=$\frac{1}{V} = x_1 + x_2 + x_3$
Subject to,

$$3x_1 + x_2 - 2x_3 \geq 1$$

$$-4x_1 - 3x_2 + 4x_3 \geq 1$$
$$2x_1 - 7x_2 + 7x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

equation (1)

The corresponding LPP for B's payoff matrix is

maximize W=$V = y_1 + y_2 + y_3$

Subject to,

$$3y_1 - 4y_2 + 2y_3 \leq 1$$
$$y_1 - 3y_2 - 7y_3 \leq 1$$
$$-2y_1 + 4y_2 + 7y_3 \leq 1$$
$$y_1, y_2, y_3 \geq 0$$

equation (2)

The standard form of the LPP maximize W=$\frac{1}{V} = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to,

$$3y_1 - 4y_2 + 2y_3 + s_1 = 1$$
$$y_1 - 3y_2 - 7y_3 + s_2 = 1$$
$$-2y_1 + 4y_2 + 7y_3 + s_3 = 1$$

Initial table:

		c_j	**1**	**1**	**1**	**0**	**0**	**0**	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3	**Ratio**
0	s_1	**1**	**3**	**-4**	**2**	**1**	**0**	**0**	**1/3**
0	s_2	**1**	**1**	**-3**	**-7**	**0**	**1**	**0**	**1**
0	s_3	**1**	**-2**	**4**	**7**	**0**	**0**	**1**	**1/2**
	Z_j		**0**	**0**	**0**	**0**	**0**	**0**	
	$Z_j - C_j$		**-1**	**-1**	**-1**	**0**	**0**	**0**	

s_1 leaves the basis

y_1 enter the basis

Iteration 1:

		c_j	**1**	**1**	**1**	**0**	**0**	**0**	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3	**Ratio**
1	y_1	**1/3**	**1**	**-4/3**	**2/3**	**1/3**	**0**	**0**	**-**
0	s_2	**2/3**	**0**	**-5/3**	**-23/3**	**-1/3**	**1**	**0**	**-**
0	s_3	**5/3**	**0**	**4/3**	**25/3**	**2/3**	**0**	**1**	**1/4**
	Z_j	**1/3**	**1**	**-4/3**	**2/3**	**1/3**	**0**	**0**	
	$Z_j - C_j$		**0**	**-7/3**	**-1/3**	**1/3**	**0**	**0**	

y_2 enter the basis

s_3 leave the basis

Iteration 2:

		c_j	**1**	**1**	**1**	**0**	**0**	**0**
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	s_3
1	y_1	**2**	**1**	**0**	**9**	**1**	**0**	**1**
0	s_2	**33/12**	**-**	**-**	**33/12**	**1/2**	**1**	**5/4**

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1	y_2	**5/4**	**0**	**1**	**25/4**	**1/2**	**0**	**3/4**
	Z_j	**13/4**	**1**	**1**	**61/4**	**3/2**	**0**	**7/4**
	$Z_j - C_j$		**0**	**0**	**57/4**	**3/2**	**0**	**7/4**

Since all $Z_j - C_j \geq 0$

The optimal solution is obtained

We have $y_1 = 2, y_2 = 5/4, y_3 = 0$

$$x_1 = 3/2, x_2 = 0, x_3 = 7/4$$

$$maxW = \frac{1}{V} = 2 + 0 + 5/4 = 13/4$$

$$\frac{1}{V} = 13/4$$

$$V = 4/13$$

$$y_1 = q_1/V \Rightarrow q_1 = y_1.V = 8/13$$

$$q_2 = y_2.V = 5/13$$

$$q_3 = y_3.V = 0$$

$$x_1 = p_1/V \Rightarrow p_1 = x_1.V = 6/13$$

$$p_2 = x_2.V = 0$$

$$p_3 = x_3.V = 7/13$$

The strategies of A and B are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 6/13 & 0 & 7/13 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 8/13 & 5/13 & 0 \end{pmatrix}.$$

9.4 Check your progress

- Define a linear programming solution of games
- Explain the concept of linear programming solution of games
- Write the method of solution of linear programming problem

9.5 Summary

For the given $m \times n$ payoff matrix we assume

$$S_A = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$$

and $S_B = \begin{pmatrix} B_1 & B_2 & \dots & B_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$

Using maximin criterion for A, we get,

$$a_{11}p_1 + a_{12}p_2 + \dots + a_{m1}p_m \geq V$$

$$a_{21}p_1 + a_{22}p_2 + \dots + a_{m2}p_m \geq V$$

.

.

.

$$a_{1n}p_1 + a_{2n}p_2 + \dots + a_{mn}p_m \geq V$$

where $p_1 + p_2 + \dots + p_m = 1, p_i \geq 0$

Dividing both side by V, we get

$$a_{11}\frac{p_1}{V} + a_{12}\frac{p_2}{V} + \dots + a_{m1}\frac{p_m}{V} \geq 1$$

$$a_{21}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \dots + a_{m2}\frac{p_m}{V} \geq 1$$

.

NOTES

. $$a_{1n}\frac{p_1}{V}+a_{2n}\frac{p_2}{V}+\ldots+a_{mn}\frac{p_m}{V}\geq 1$$

$$\frac{p_1}{V}+\frac{p_2}{V}+\ldots+\frac{p_m}{V}=\frac{1}{V}$$

Denote $\frac{p_1}{V},\frac{p_2}{V},\ldots,\frac{p_m}{V}$ by
$x_1,x_2,\ldots x_m$ we get the LPP

$$max\frac{1}{V}=Z=x_1+x_2+\ldots+x_m$$

$$(maxV \Leftrightarrow min\frac{1}{V})$$

Subject to,

$$a_{11}x_1+a_{21}x_2+\ldots+a_{m1}x_m\geq V$$
$$a_{21}x_1+a_{22}x_2+\ldots+a_{m2}x_m\geq V$$

.
.
.

$a_{1n}x_1+a_{2n}x_2+\ldots+a_{mn}c_m\geq V$ equation (1)

$$x_1,x_2,\ldots x_m\geq 0$$

Similarly, considering minimax criterion for A, we get

$$a_{11}\frac{q_1}{V}+a_{12}\frac{q_2}{V}+\ldots+a_{1n}\frac{q_n}{V}\leq 1$$
$$a_{21}\frac{q_1}{V}+a_{22}\frac{q_2}{V}+\ldots+a_{2n}\frac{q_n}{V}\leq 1$$

.
.
.

$$a_{m1}\frac{q_1}{V}+a_{m2}\frac{q_2}{V}+\ldots+a_{mn}\frac{q_n}{V}\leq 1$$
$$\frac{q_1}{V}+\frac{q_2}{V}+\ldots+\frac{q_n}{V}=\frac{1}{V}$$

Denote $\frac{q_1}{V},\frac{q_2}{V},\ldots,\frac{q_n}{V}$ by $y_1,y_2,\ldots y_n$
we get the LPP

$$max\frac{1}{V}=W=y_1+y_2+\ldots+y_n$$

$$(minV \Leftrightarrow max\frac{1}{V})$$

Subject to,

$$a_{11}y_1+a_{12}y_2+\ldots+a_{1n}y_n\leq V$$
$$a_{21}y_1+a_{22}y_2+\ldots+a_{2n}y_n\leq V$$

.
.
.

$a_{m1}y_1+a_{m2}y_2+\ldots+a_{mn}y_n\geq V$ equation (1)

$$y_1,y_2,\ldots y_n\leq 0$$

9.6 Keywords

Optimal strategy:

For the given $m\times n$ payoff matrix we assume

$$S_A=\begin{pmatrix}A_1 & A_2 & \ldots & A_m\\ p_1 & p_2 & \ldots & p_m\end{pmatrix}$$

and $$S_B=\begin{pmatrix}B_1 & B_2 & \ldots & B_n\\ q_1 & q_2 & \ldots & q_n\end{pmatrix}$$

NOTES

9.7 Self Assessment Questions and Exercises

1. Solve the following rectangular game graphically.

Player B

Player A $\begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix}$

2. Solve the following games graphically. The pay-off is for player A.

Player B

Player A $\begin{pmatrix} 2 & -1 & 5 & -2 & 6 \\ -2 & 4 & -3 & 1 & 0 \end{pmatrix}$

3. Solve the following game by simpler method.

Player B

Player A $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$

4. Solve (3× 3) game by the simplex method of linear programming whose pay-off is given as follows:

Player B

3	-1	-3
-3	3	-1
-4	-3	3

Player A

9.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi, 2008.

UNIT-X CLASSICAL OPTIMIZATION THEORY

NOTES

Structure

10.1 Introduction

Classical optimization theory uses differential calculus to determine points of maxima and minima for unconstraint and constraint functions.
The methods may not be suitable for efficient numerical computations, but underline the theory provides the most basis for non-linear programming algorithm. In this chapter we discuss about the necessary and sufficient conditions for determining the unconstraint extreme, the Jacobian and lagrangian method for problems with equality conditions (constraint), and the Karush-Kuhn-tucker (KKT) conditions for problems with inequality condition. The KKT method provides the most unifying theory for all non-linear programming problems.

10.2 Objectives

After going through this unit, you will be able to:

- Define stationary point
- Define weak and strong maximum
- Define inflection and saddle point
- Understand the concept of extrema

10.3 Classical Optimization theory-Jacobian Method

Unconstraint problems:

An extreme point of a function f(X) defines either a maximum or a minimum of the function

(ie) A point $x_0 = (x_1^0, x_2^0, \ldots x_j^0, \ldots x_n^0)$ is a maximum if $f(x_0 + h) \leq f(x_0)$ for all $h = h_1, h_2, \ldots h_j, \ldots h_n$ where $|h_j|$ is sufficiently small for all j.

[OR]

x_0 is a maximum if the values of f at every point in the neighbourhood of x_0 does not exceed $f(x_0)$.

Similarly x_0 is a minimum if $f(x_0 + h) \geq f(x_0)$

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In the above figure the maxima and minima of a single variable function $f(x)$ over the interval [a,b] the points $x_1, x_2, x_3, x_4 and x_6$ are all extrema of f(x) with $x_1, x_3 and x_6$ are maxima and $x_2 and x_4$ as minima

$$f(x_6) = maximum f(x_1), f(x_3), f(x_6)$$

Here $f(x_6)$ is a global are absolute maximum and $f(x_1)$ and $f(x_3)$ are local or relative maximum.

Similarly $f(x_4)$ is the local minimum and $f(x_2)$ is the global minimum.

Although x_1 is a maximum point, it differs from the remaining local maxima in that the value of f corresponding to atleast one point in the neighbourhood of x_1 equals $f(x_1)$.

Here $f(x_1)$ is a weak maximum whereas x_3 and x_6 are strong maximum.

In general for h as defined earlier x_0 is a weak maximum if $f(x_0 + h) \leq f(x_0)$ and a strong maximum if $f(x_0 + h) < f(x_0)$

Necessary and Sufficient Conditions:

A necessary condition for x_0 to be an extreme point of f(x) is $\nabla f(x_0) = 0$

The sufficient condition for a stationary point x_0 to be an extreme point is that the hessian matrix H evaluated as x_0 satisfying the following conditions.

a) H is positive definite if x_0 is a minimum point.

b) H is negative definite if x_0 is a maximum point.

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 . \partial x_2} & \frac{\partial^2 f}{\partial x_1 . \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 . \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 . \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 . \partial x_1} & \frac{\partial^2 f}{\partial x_3 . \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix}$$

1. Examine the following functions for extreme points.

$$f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$$

Solution:

$\frac{\partial f}{\partial x_1} = 1 - 2x_1, \frac{\partial f}{\partial x_2} = x_3 - 2x_2$

$$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3$$

x_0 to be a extrema we have $\nabla f(x_0) = 0$

$$1 - 2x_1 = 0, \quad x_3 - 2x_2 = 0, \quad 2 + x_2 - 2x_3 = 0$$

$$1 - 2x_1 = 0 \Rightarrow x_1 = 1/2$$

$$2x_3 - 4x_2 = 0$$

$$-2x_3 + x_2 + 2 = 0$$

$$-3x_2 = -2x_2 = 2/3$$

$$x_3 - 2x_2 = 0$$

$$x_3 = 4/3$$

$$X_0 = (1/2, 2/3, 4/3)$$

$\frac{\partial^2 f}{\partial x_1^2} = -2, \frac{\partial^2 f}{\partial x_2^2} = -2, \frac{\partial^2 f}{\partial x_3^2} = -2, \frac{\partial^2 f}{\partial x_1 . \partial x_2} = 0, \frac{\partial^2 f}{\partial x_1 . \partial x_3} = 0,$

$\frac{\partial^2 f}{\partial x_2 . \partial x_1} = 0, \frac{\partial^2 f}{\partial x_2 . \partial x_3} = 1, \frac{\partial^2 f}{\partial x_3 . \partial x_1} = 0, \quad \frac{\partial^2 f}{\partial x_3 . \partial x_2} = 1$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 . \partial x_2} & \frac{\partial^2 f}{\partial x_1 . \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 . \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 . \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 . \partial x_1} & \frac{\partial^2 f}{\partial x_3 . \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix}$$

$$H = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$D_1 = |-2|, D_2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = 4,$$

$$D_3 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} = -6$$

The principle minus of H have the values -2,4,-6

H is a negative definite and the extreme point $x_0 = (1/2, 2/3, 4/3)$ is a maximum point.

2. $f_x = x_1^2 - 12x_1 + x_2^2 - 8x_2 + x_3^2 - 4x_3$

Solution:

$\frac{\partial f}{\partial x_1} = -12 + 12x_1, \frac{\partial f}{\partial x_2} = 2x_2 - 8$

$$\frac{\partial f}{\partial x_3} = 2x_3 - 4$$

x_0 to be a extrema we have $\nabla f(x_0) = 0$

$2x_1 - 12 = 0, 2x_2 - 8 = 0, 2x_3 - 4 = 0$

$$2x_1 - 12 = 0 \Rightarrow x_1 = 6$$

$2x_2 - 8 = 0 \Rightarrow 2x_2 = 8 \Rightarrow x_2 = 4$

$2x_3 - 4 = 0 \Rightarrow x_3 = 2$

$X_0 = (6,4,2)$

$\frac{\partial^2 f}{\partial x_1^2} = 2, \quad \frac{\partial^2 f}{\partial x_2^2} = 2, \quad \frac{\partial^2 f}{\partial x_3^2} = 2, \quad \frac{\partial^2 f}{\partial x_1 . \partial x_2} = 0$

$\frac{\partial^2 f}{\partial x_1 . \partial x_3} = 0, \frac{\partial^2 f}{\partial x_2 . \partial x_1} = 0, \frac{\partial^2 f}{\partial x_2 . \partial x_3} = 0, \frac{\partial^2 f}{\partial x_3 . \partial x_1} = 0, \frac{\partial^2 f}{\partial x_3 . \partial x_2} = 0$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 . \partial x_2} & \frac{\partial^2 f}{\partial x_1 . \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 . \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 . \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 . \partial x_1} & \frac{\partial^2 f}{\partial x_3 . \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix}$$

NOTES

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$D_1 = |2| = 2, D_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4, \qquad D_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 8$$

The principle plus of H have the values 2,4,6
H is a positive definite and the extreme point $x_0 = (6,4,2)$ is a minimum point.
A necessary condition for x_0 to a extreme point of f(x) is $\nabla f(x) = 0$
The sufficient to the single variable function $f(x)$ is as follows.
Given that y_0 is the stationary point then
a) y_0 is a maximum if $f^{''} y_0 < 0$
b) y_0 is a minimum if $f^{''} y_0 > 0$
c) If $f^{''} y_0 = 0$ then the higher order derivatives must be investigated as follows.
Given y_0 is a stationary point of f(y), if the first n-1 derivatives are zero and $f^n y_0 \neq 0$ then
a) If n is odd, y_0 is an inflection point
b) If n is even, y_0 is a minimum point if $f^n y_0 \geq 0$ and a maximum point if $f^n y_0 \leq 0$
3. Solve the function $f(y) = y^4$
Solution:
$f^{'}(y) = 4y^3, \qquad f^{''}(y) = 12y^2, \qquad f^{'''}(y) = 24y, \qquad f^4(y) = 24$
x_0 to a extreme point of f(x) is $\nabla f(x) = 0$

$$y^3 = 0 \Rightarrow y = 0$$

y=0 is a stationary point.
when y=0
$f^{'}(y) = 0, f^{''}(y) = 0, \qquad f^{'''}(y) = 0, \qquad f^4(y) = 24$
Here n=4 (even) and $f^4(y) = 24$
y_0 is a minimum point.

4. $f(y) = y^3$

Solution:
$f^{'}(y) = 3y^2, f^{''}(y) = 6y, \ f^{'''}(y) = 6$
x_0 to a extreme point of f(x) is $\nabla f(x) = 0$

$$3y^2 = 0 \Rightarrow y = 0$$

y=0 is a stationary point.
when y=0

$$f^{'}(y) = 0, f^{''}(y) = 0, f^{'''}(y) = 6$$

Here n=3 (odd) and $f^{'''}(y) = 0$
y_0 is a inflection point.

5. Examine the following function are extreme point.
$[OR]$ Find the extreme points for the following functions. $f(x) = x^3 + x$
Solution:

$$f^{'}(x) = 3x^2 + 1$$
$$f^{''}(x) = 6x$$

$$f'''(x) = 6$$

To find the extreme points we have $\nabla f(x_0) = 0$

$$3x^2 + 1 = 0$$
$$x = \pm i\sqrt{1/3}$$
$$f''(x) = 6\sqrt{1/3}$$

y_0 is minimum when $x = -i\sqrt{1/3}$

y_0 is maximum

6. $f(x) = x^4 + x^2$

Solution:

$$f'(x) = 4x^3 + 2x$$
$$f''(x) = 12x^2 + 2$$
$$f'''(x) = 24x$$
$$f^4(x) = 24$$

To find the extreme points we have $\nabla f(x_0) = 0$

$$4x^3 + 2x = 0$$
$$x = \pm i\sqrt{1/2}$$

y_0 is minimum

$$f''(x) = 12(i\sqrt{1/2})^2 + 2 = -4$$

y_0 is maximum

$$f''(x) = 12(i\sqrt{1/2})^2 + 2 = -4$$

y_0 is maximum

7. $f(x) = 6x^5 - 4x^3 + 10$

Solution:

$$f'(x) = 30x^4 - 12x^2$$
$$f''(x) = 120x^3 - 24x$$
$$f'''(x) = 360x^2 - 24$$
$$f^4(x) = 720x$$
$$f^5(x) = 720$$

To find the extreme points we have $\nabla f(x_0) = 0$

$$30x^4 - 12x^2 = 0$$
$$x = \pm\sqrt{2/5}$$
$$f''(x) = 120x^3 - 24x = 0$$

Here $f''(y_0) = 0$

$$f'''(y_0) = f'''(0) = -24$$

n=3 and $f'''(0) < 0$

when x=0, y_0 is maximum point.

when x=0, y_0 is maximum point.

$$f''(\sqrt{2/5}) = 24\sqrt{2/5}$$

y_0 is minimum point.

$$f''(-\sqrt{2/5}) = -24\sqrt{2/5}$$

y_0 is maximum point.

8. Examine the following functions are extreme point.

$$f(x) = 4x_1^2 + 8x^1x_2$$

Solution:

$\frac{\partial f}{\partial x_1} = 8x_1 + 8x_2, \frac{\partial f}{\partial x_2} = 8x_1$

x_0 to be a extrema we have $\nabla f(x_0) = 0$

$$8x_1 + 8x_2 = 0$$
$$8x_1 = 0$$
$$x_1 = 0$$
$$8(0) + 8x_2 = 0$$
$$8x_2 = 0$$
$$X_0 = (0,0)$$

$\frac{\partial^2 f}{\partial x_1^2} = 8, \frac{\partial^2 f}{\partial x_2^2} = 0, \frac{\partial^2 f}{\partial x_1.\partial x_2} = 8, \frac{\partial^2 f}{\partial x_2.\partial x_1} = 8$

$$H = \begin{pmatrix} 8 & 8 \\ 8 & 0 \end{pmatrix}$$

$$D_1 = |8| = 8, \qquad D_2 = \begin{pmatrix} 8 & 8 \\ 8 & 0 \end{pmatrix} = -64$$

The principle minus of H have the values 8,-64.

H is a negative definite and the extreme point $X_0 = (0,0)$ is a maximum point.

Constraint problem:

This section deals with the optimization of constraint functions.

Equality Constraint:

This section presents two methods:

a) Jacobian method

b) Lagrangian method

The Lagrangian method can be derived logically from the Jacobian method.

Both methods are used to solve the constraint problems.

Jacobian method:(Algorithm)

Consider the problem minimize Z=f(x) subject to g(x)=0, where $x = (x_1, x_2, \ldots x_n)$ and $g = (g_1, g_2, \ldots g_m)^T$ the function f(x) and g(x) are continuous

By tailors theorem,

For $x + \Delta x$ is the feasible neighbourhood of x, we have

$$f(x + \Delta x) - f(x) = \nabla f(x) \Delta x + 0(\Delta x_j^2)$$

and $g(x + \Delta x) - g(x) = \nabla g(x) \Delta x + 0(\Delta x_j^2)$

As $\Delta x_j \to 0$

The equations reduces to

$\partial f(x) = \nabla f(x) dx$ and $\partial g(x) = \nabla g(x) dx$

Define $x = (y, z)$ such that $y = (y_1, y_2, \ldots y_m)$ and $z = (z_1, z_2, \ldots z_{n-m})$

The vectors y and z are called the dependent and independent variables respectively.

The gradient vectors of "f and g" interms of y and z, we get

$$\nabla f(y, z) = (\nabla_y f, \nabla_z f)$$
$$\nabla g(y, z) = (\nabla_y g, \nabla_z g)$$

Then define, J=$\nabla_y g = \begin{pmatrix} \nabla_y g_1 \\ \nabla_y g_2 \\ . \\ . \\ . \\ \nabla_y g_m \end{pmatrix}$

and C=$\nabla_z g = \begin{pmatrix} \nabla_z g_1 \\ \nabla_z g_2 \\ . \\ . \\ . \\ \nabla_z g_m \end{pmatrix}$

Where 'J' is called Jacobian matrix and 'c' is called the control matrix.

J is assumed to be non-singular.

The original set of equations $\partial f(x)$ and ∂x may be written as

$$\partial f(y,z) = \nabla_y f \partial y + \nabla_z f \partial z$$

and $J\partial y = -C\partial z$

J is non-singular.J^{-1} exists

Hence $\partial y = -J^{-1} C \partial z$

substituting ∂y in the equations for $\partial f(x)$

$$\partial f(y,z) = \nabla_y f \partial z - \nabla_y f J^{-1} C \partial z$$

=$(\nabla_z f - \nabla_y f J^{-1} C)\partial z$

From this equation the constraint derivative w.r.to the independent vector z is given by

$$\nabla_c f = \partial c f(y,z)/\partial c^z = \nabla_z f - \nabla_y f J^{-1} c$$

where $\nabla_c f$ is the constraint gradient vector of f w.r.to z

Jacobian method:

13. Solve minimum $(f(x)) = x_1^2 + x_2^2 + x_3^2$

subject to the constraints,

$$g_1(x) = x_1 + x_2 + 3x_3 - 2 = 0$$

$$g_2(x) = 5x_1 + 2x_2 + x_3 - 5 = 0$$

Using Jacobian method:

Solution:

Let Y=(x_1, x_2) and Z=x_3

$$\nabla_y f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})$$

=$(2x_1, 2x_2)$

$$\nabla_z f = (\frac{\partial f}{\partial x_1})$$

=$2x_3$

NOTES

We know that $J=\nabla_y g = \begin{pmatrix} \nabla_y g_1 \\ \nabla_y g_2 \\ . \\ . \\ . \\ \nabla_y g_m \end{pmatrix}$

(ie) $J=\nabla_y g = \begin{pmatrix} \nabla_y g_1 \\ \nabla_y g_2 \end{pmatrix}$

$J=\nabla_y g = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix}$

Adj $J=\nabla_y g = \begin{pmatrix} 2 & -1 \\ -5 & 1 \end{pmatrix}$, $|J|=\nabla_y g = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix} = 3$

$$J^{-1} = \frac{1}{|J|} AdjJ$$

$=\frac{1}{-3}\begin{pmatrix} 2 & -1 \\ -5 & 1 \end{pmatrix} = J^{-1}$

$J=\nabla_z g = \begin{pmatrix} \nabla_z g_1 \\ \nabla_z g_2 \end{pmatrix} = J = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

We know that,

$$\nabla_c f = \partial c f(y,z)/\partial c^z = \nabla_z f - \nabla_y f J^{-1} c$$

$=2x_3 - (2x_1, 2x_2)\begin{pmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$=2x_3 - (2x_1, 2x_2)\begin{pmatrix} -6/3 + 1/3 \\ 15/3 - 1/3 \end{pmatrix}$

$=2x_3 - (2x_1, 2x_2)\begin{pmatrix} -5/3 \\ 14/3 \end{pmatrix}$

$=2/3(3x_3 + 5x_1 - 14x_2)$

$$\nabla_c f = 2/3(5x_1 - 14x_2 + 3x_3)$$

To find the stationary points, we have $\nabla_c f = 0$, $g_1(x) = 0$, $g_2(x) = 0$

$$\nabla_c f = 2/3(5x_1 - 14x_2 + 3x_3) = 0$$

$=10x_1 - 28x_2 + 6x_3 = 0$ equation (1)

$g_1(x) = 0 \Rightarrow x_1 + x_2 + 3x_3 = 2$ equation (2)

$g_2(x) = 0 \Rightarrow 5x_1 + 2x_2 + x_3 = 5$ equation (3)

These equation can be written as,

$\begin{pmatrix} 10 & -28 & 6 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 & -28 & 6 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$

Let $A=\begin{pmatrix} 10 & -28 & 6 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix}$

To find: Adj A

cofactor $a_{11} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = 1 - 6 = -5$

$$a_{12} = -\begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix} = 14$$

$$a_{13} = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix} = -3$$

$$a_{21} = -\begin{pmatrix} -28 & 6 \\ 2 & 1 \end{pmatrix} = 40$$

$$a_{22} = \begin{pmatrix} 10 & 6 \\ 5 & 1 \end{pmatrix} = -20$$
$$a_{23} = \begin{pmatrix} 10 & -28 \\ 5 & 2 \end{pmatrix} = -160$$
$$a_{31} = \begin{pmatrix} -28 & 6 \\ 1 & 3 \end{pmatrix} = -90$$
$$a_{32} = -\begin{pmatrix} 10 & 6 \\ 1 & 3 \end{pmatrix} = -24$$
$$a_{33} = \begin{pmatrix} 10 & -28 \\ 1 & 1 \end{pmatrix} = 38$$

$$\text{Adj A}=\begin{pmatrix} -5 & 40 & -90 \\ 14 & -20 & -24 \\ -3 & -160 & 38 \end{pmatrix}$$

$$|A| = \begin{pmatrix} 10 & -28 & 6 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix} = -460$$

$$A^{-1} = 1/-460.\begin{pmatrix} -5 & 40 & -90 \\ 14 & -20 & -24 \\ -3 & -160 & 38 \end{pmatrix}$$

$$=\begin{pmatrix} 1/92 & -2/23 & 9/46 \\ 7/230 & 1/23 & 12/230 \\ 3/460 & 8/23 & -19/230 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1/92 & -2/23 & 9/46 \\ 7/230 & 1/23 & 12/230 \\ 3/460 & 8/23 & -19/230 \end{pmatrix}.\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0.80 \\ 0.35 \\ 0.28 \end{pmatrix}$$

10.4 Check your progress

- Define stationary point
- Define weak and strong maximum
- Define inflection and saddle point
- Understand the concept of extrema

10.5 Summary

- An extreme point of a function f(X) defines either a maximum or a minimum of the function

  (ie) A point $x_0 = (x_1^0, x_2^0, \ldots x_j^0, \ldots x_n^0)$ is a maximum if $f(x_0 + h) \leq f(x_0)$ for all $h = h_1, h_2, \ldots h_j, \ldots h_n$ where $|h_j|$ is sufficiently small for all j.

  [OR]

  x_0 is a maximum if the values of f at every point in the neighbourhood of x_0 does not exceed $f(x_0)$.
- Similarly x_0 is a minimum if $f(x_0 + h) \geq f(x_0)$
- A necessary condition for x_0 to be an extreme point of f(x) is $\nabla f(x_0) = 0$
- The sufficient condition for a stationary point x_0 to be an extreme point is that the hessian matrix H evaluated as x_0 satisfying the following conditions.

  a) H is positive definite if x_0 is a minimum point.

  b) H is negative definite if x_0 is a maximum point.

NOTES

10.6 Keywords

Maximum Point:

A point $x_0 = (x_1^0, x_2^0, \dots x_j^0, \dots x_n^0)$ is a maximum if $f(x_0 + h) \leq f(x_0)$ for all $h = h_1, h_2, \dots h_j, \dots h_n$ where $|h_j|$ is sufficiently small for all j.

[OR]

x_0 is a maximum if the values of f at every point in the neighbourhood of x_0 does not exceed $f(x_0)$.

10.7 Self Assessment Questions and Exercises

1. Examine the following functions are extreme points.
   a) $f(x) = x^3+x$
   b) $f(x) = x^4+x^2$
   c) $f(x) = 4x^2_1+8x_1x_2$
   d) $f(x) = 900x_1+900x_2 -10x^2_1+10x_1x_2-10x^2_2$
   e) $f(x) = 6x^5-4x^3+10$
2. Examine the following functions for extreme points.
   a) $f(x) = x_1+2x_3 -x^2_1+x_2x_3 - x^2_2 - x^2_3$
   b) $f(x) = -12x_1-8x_2-4x_3 +x^2_1+x^2_2 - 4x^2_3$
3. Verify that the function
   $f(x_1,x_2,x_3) = 2x_1x_2x_3-4x_1x_3-2x_2x_3+x^2_1+x^2_2+x^2_3-2x_1-4x_2+4x_3$
   has the stationary points (0,3,1), (0,1,-1), (1,2,0), (2,1,1) and (2,3,-1). Use the sufficiency condition to identify the extreme points.
4. Solve the following simultaneous equations by converting the system to a nonlinear objective function with no constraints.
   $X_2 -x^2_1 = 0$
   $X_2 – x_1 = 2$
5. Solve by the Jacobian method:
   Minimize $f(X) = \sum_{i=1}^{n} {x_i}^2$

10.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi,2008.

UNIT- XI LAGRANGIAN AND NEWTON RAPHSON METHODS

Structure

11.1 Introduction

This chapter deals with Lagrangian and Newton Raphson methods. The definition of lagrangian multipliers, necessary and sufficient conditions are also discussed. Also some theorems and problems are also given.

11.2 Objectives

After going through this unit, you will be able to:

- Define lagrangian multipliers
- Understand the concept of Lagrangian and Newton Raphson methods
- Solve the problems in Lagrangian and Newton Raphson methods

11.3 Lagrangian and Newton Raphson Methods

Lagrangian method:

The equation $\partial f - \lambda \partial y = 0$, this equation satisfy the necessary condition for stationary points because $\frac{\partial f}{\partial g}$ is computed such that $\nabla_c f = 0$

A move convenient from for presenting this equation is to take the partial derivatives w.r.to all x_j.

Thus $\frac{\partial}{\partial x_j}(f - \lambda g) = 0, j = 1,2, \ldots n$

The resulting equation together constrained equations $g(x) = 0$ gives the feasible values of x and λ that satisfy the necessary condition for stationary points.

The given procedure defines, the lagrangian method for identifying the stationary points of optimization problem, with equality constraints.

Let $L(x, \lambda) = f(x) - g(x)$, the function "L" is called the Lagrangian function and the parameter λ the lagrangian multipliers the equation $\frac{\partial L}{\partial \lambda} = 0$, $\frac{\partial L}{\partial x} = 0$, gives the necessary condition for determining stationary point of f(x) subject to $g(x) = 0$

Sufficient condition for the lagrangian method exist but the computation are to hard to find.

Newton Raphson method:

In general the necessary condition equations $\nabla f(x) = 0$ may be difficult to solve numerically.

Newton Raphson method is an iterative procedure for solving non-linear equations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1. Solve by using newtons method find the root between 0 and 1 of $x^3 - 6x + 4 = 0$ correct to 5 decimal places.

Solution:

$$f(x) = x^3 - 6x + 4$$
$$f'(x) = 3x^2 - 6$$

Given that the root lies between 0 and 1.

Let $x_0 = 0.5$

Iteration 1:

n=0, $x_0 = 0.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$f(x_0) = x_0^3 - 6x_0 + 4$$
$$f(0.5) = (0.5)^3 - 6(0.5) + 4 = 1.125$$
$$f'(x_0) = 3x_0^2 - 6 = 3(0.5)^2 - 6 = -5.24$$
$$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.71429$$

Iteration 2:

n=1, $x_1 = 0.71429$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$f(0.71429) = (0.71429)^3 - 6(0.71429) + 4 = 0.07866$$
$$f'(0.71429) = 3(0.71429)^2 - 6 = -4.4694$$
$$x_2 = 0.71429 - \frac{f(0.71429)}{f'(0.71429)} = 0.73189$$
$$x_2 = 0.7319$$

Iteration 3:

n=2, $x_2 = 0.7319$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$f(0.7319) = (0.7319)^3 - 6(0.7319) + 4 = 0.0007$$
$$f'(0.7319) = 3(0.7319)^2 - 6 = -4.3930$$
$$x_3 = 0.7319 - \frac{f(0.7319)}{f'(0.7319)} = 0.7321$$

Iteration 4:

n=3, $x_3 = 0.7321$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

NOTES

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$f(0.7321) = (0.7321)^3 - 6(0.7321) + 4 = -0.00022$$
$$f'(0.7321) = 3(0.7321)^2 - 6 = -4.3921$$
$$x_4 = 0.7321 - \frac{f(0.7321)}{f'(0.7321)} = 0.7321$$

The root is $x_4 = 0.7321$

2. Find the roots of the function $x - \cos x = 0$ by newtons method correct to 3-decimal places.

Solution:

Let $f(x) = x - \cos x$
$$f'(x) = 1 + \sin x$$

when x=0
$$f(0) = 0 - \cos 0 = -1$$

when x=1
$$f(1) = 1 - \cos 1 = 1$$

The roots lies between 0 and 1.

Iteration 1:

n=0, $x_0 = 0.5$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$f(x_0) = x_0 - \cos x_0$$
$$f(0.5) = (0.5) - \cos(0.5) = -0.4996$$
$$f'(x_0) = 1 + \sin x_0 = 1.0087$$
$$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.9956$$

Iteration 2:

n=1, $x_1 = 0.9956$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$f(x_1) = 0.9956 - \cos(0.9956) = -0.00425$$
$$f'(x_1) = 1 + \sin(0.9956) = 1.01738$$
$$x_2 = 0.9956 - \frac{f(0.9956)}{f'(0.9956)} = 0.99978$$

Iteration 3:

n=2, $x_0 = 0.99978$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$f(x_2) = 0.99978 - \cos(0.99978) = -0.000068$$
$$f'(x_2) = 1 + \sin(0.99978) = 1.01745$$
$$x_3 = 0.99978 - \frac{f(0.99978)}{f'(0.99978)} = 0.9998$$

The roots is $x = 0.999$

11.4 Check your progress

- Define Lagrangian multiplier
- State necessary and sufficient conditions of a point to be a stationary point

11.5 Summary

- The equation $\partial f - \lambda \partial y = 0$, this equation satisfy the necessary condition for stationary points because $\frac{\partial f}{\partial g}$ is computed such that $\nabla_c f = 0$
- Let $L(x,\lambda) = f(x) - g(x)$, the function "L" is called the Lagrangian function and the parameter λ the lagrangian multipliers the equation $\frac{\partial L}{\partial \lambda} = 0$, $\frac{\partial L}{\partial x} = 0$, gives the necessary condition for determining stationary point of f(x) subject to $g(x) = 0$
- Sufficient condition for the lagrangian method exist but the computation are to hard to find.

Newton Raphson method:

In general the necessary condition equations $\nabla f(x) = 0$ may be difficult to solve numerically.

Newton Raphson method is an iterative procedure for solving non-linear equations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

11.6 Keywords

Let $L(x,\lambda) = f(x) - g(x)$, the function "L" is called the **Lagrangian function** and the parameter λ the lagrangian multipliers**.**

Newton Raphson method is an iterative procedure for solving non-linear equations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

11.7 Self Assessment Questions and Exercises

1. Use Newton Raphson to solve $f(x) = x^3 + x$
2. Solve $f(x)= x^2_1+x^2_2+x^2_3-12x_1-8x_2-4x_3$ by Newton Raphson method.
3. Obtain the necessary and sufficient condition for the optimum solution of the following non-linear programming problem
   Minimize $z = f(x_1,x_2) = 3e^{2x_1+1}+2e^{x_2+5}$
   Subject to the constraints $x_1+x_2 =7$
   $x_1,x_2 \geq 0$.
4. Solve the non-linear programming problem
   Minimize $z= 2x^2_1+2x^2_2+2x^2_3-24x_1-8x_2-12x_3+200$
   Subject to the constraints $x_1+x_2 +x_3 = 11$
   $x_1,x_2 ,x_3 \geq 0$.
5. Solve the non-linear programming problem
   Optimize $z= 4x^2_1+2x^2_2+x^2_3-4x_1x_2$
   Subject to the constraints $x_1+x_2 +x_3 = 15$, $2 x_1+x_2 +2x_3 = 20$.

11.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi,2008.

NOTES

BLOCK IV: KKT METHODS, SEPERABLE AND QUADRATIC PROGRAMMINGS

UNT XII KARUSH KUHN TUCKER CONDITIONS

Structure

12.1 Introduction

In this chapter, we studied the main role of Karush Kuhn Tucker necessary conditions for determining the stationary points. These conditions are also sufficient under certain rules also sufficient under certain rules that will be stated later Also some problems are solved.

12.2 Objectives

After going through this unit, you will be able to:

- State KKT conditions for the optimality
- Understand the importance of KKT conditions
- Solve the problems in KKT methods

12.3 The second fundamental form:

Karush-kuhn-Tucker(KKT) conditions:

This section extends the lagrangian method to problems with inequality constraints. The main contribution of the section is the development of the general Karush-Kuhn-Tucker(KKT) necessary conditions.

Necessary condition:

Consider the problem, minimize Z=f(x), $g(x) \leq 0$.

The inequality constraints may be converted into equations by using non-negative slack variable. Let $s_i^2 (\geq 0)$ be the slack variable added to the i^{th} constraint $g_i(x) = 0$ and define $s = (s_1, s_2, \dots s_m)^T$,$s^2 = (s_1^2, s_2^2, \dots s_m^2)^T$

where m is the total number of inequality constraints. The Lagrangian function is given by

$$L(X, S, \lambda) = f(x) - \lambda[g(x) + s^2]$$

Given the constraints $g(x) \leq 0$ a necessary conditions for optimality is that λ be non-negative (non-positive) for maximization(minimization) problems.

This result is justified by, nothing that the vector λ measures the rate of variation of f with respect to g that is $\lambda = \partial f / \partial g$

In the maximization case, as the right hand side of the constraint $g(x) \leq 0$ increases from 0 to the vector ∂g and $\lambda \geq 0$.

Similarly, for minimization, as the right hand side of the constraints increases, f cannot increases, which implies that $\lambda \leq 0$

If the constraints are equalities that is $g(x) = 0$, then λ becomes unrestricted in sign.

The restrictions on λ holds as part of the remaining conditions will now be developed.

Taking the partial derivatives as L w.r.to X,s and λ we obtain

$$\frac{\partial L}{\partial x} = \nabla f(x) - \lambda \nabla g(x) = 0$$

$$\frac{\partial L}{\partial s_i} = -2\lambda_i s_i = 0, i = 1,2,\ldots m$$

$$\frac{\partial L}{\partial \lambda} = -(g(x) + s^2) = 0$$

The second set of equations reveals the following results.

If $\lambda_i \neq 0$, then $s_i^2 = 0$, which means that the corresponding resource is scarce, and hence it is consumed completely (equality constraint.)

If $s_i^2 > 0$, then $\lambda_i = 0$. This means resource i is not scarce and consequently it has no affect on the value of f

(ie) $\lambda_i = \frac{\partial f}{\partial g_i} = 0$

From the second and third sets of equations, we obtain $\lambda_i g_i(x) = 0$

The KKT necessary conditions for maximization problems are summarized as $\lambda \geq 0$

$$\nabla f(x) - \lambda \nabla g(x) = 0$$

$\lambda_i g_i(x) = 0$, i=1,2,...n

$$g(x) \leq 0$$

These conditions apply to the minimization case as well, except that λ must be non-positive.

Sufficiency of the KKT condition:

The Kuhn-Tucker necessary conditions are also sufficient if the objective function and the solution space satisfy specific conditions.

These conditions are summarized in the table as follows.

We define the generalized non-linear problems as,

maximize or minimize Z=f(x)

Subject to, $g_i(x) \leq 0, i = 1,2,\ldots r$

$$g_i(x) \geq 0, i = r+1,\ldots p$$

$$g_i(x) \neq 0, i = p+1,\ldots m$$

$$L(X,S,\lambda) = f(x) - \Sigma_{i=1}^{r}\lambda_i[g_i(x) + s_i^2] - \sum_{i=r+1}^{p}\lambda_i[g_i(x) - s_i^2] - \sum_{i=p+1}^{m}\lambda_i g_i(x)$$

where λ_i is the lagrangian multiplier associated with constraint i.

1. Consider the following minimization problem

Minimize f(X) = ${x_1}^2 + {x_2}^2 + {x_3}^2$

Subject to

g_1(X) = 2 $x_1 + x_2$ -5 ≤ 0

g_2(X) = $x_1 + x_3$ -2 ≤ 0

g_3(X) = 1- x_1 -5 ≤ 0
g_4(X) = 2 - x_2 ≤ 0
g_5(X) = -x_3 ≤ 0
This is a minimization problem, hence **λ**≤ 0.

The KKT conditions are thus given as
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \leq 0$.

$$(2\,x_1, 2\,x_2, 2\,x_3)-(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

$\lambda_1\, g_1 = \lambda_2\, g_2 = ... = \lambda_5\, g_5 = 0$
g(X) ≤ 0.
These conditions reduce to $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \leq 0$.
$2\,x_1 - 2\,\lambda_1 - \lambda_2 + \lambda_3 = 0$
$2\,x_2 - 2\,\lambda_1 + \lambda_4 = 0$
$2\,x_3 - 2\,\lambda_2 + \lambda_5 = 0$
$\lambda_1(2\,x_1 + x_2 - 5) = 0$
$\lambda_2(x_1 + x_3 - 2) = 0$
$\lambda_3(1 - x_1) = 0$
$\lambda_4(2 - x_2) = 0$
$\lambda_5 x_3 = 0$

$2\,x_1 + x_2 \leq 5$
$x_1 + x_3 \leq 2$
$x_1 \geq 1, \quad x_2 \geq 2, \ x_3 \geq 0$
The solution is $x_1 = 1, \quad x_2 = 2, \ x_3 = 0, \ \lambda_1 = \lambda_2 = \lambda_5 = 0, \lambda_3 = -2, \lambda_4 = -4$.

Because both f(X) and the solution g(X))≤ 0 are convex, L(X,S,**λ**) must be convex and the resulting stationary point yields a global constrained minimum.

12.4 Check your progress

- State KKT conditions
- Explain the importance of KKT conditions

12.5 Summary

Consider the problem,
minimize Z=f(x)

$$g(x) \leq 0$$

The inequality constraints may be converted into equations by using non-negative slack variable. Let $s_i^2 (\geq 0)$ be the slack variable added to the i^{th} constraint $g_i(x) = 0$ and define $s = (s_1, s_2, \dots s_m)^T, s^2 = (s_1^2, s_2^2, \dots s_m^2)^T$

where m is the total number of inequality constraints. The Lagrangian function is given by

$$L(X, S, \lambda) = f(x) - \lambda[g(x) + s^2]$$

Given the constraints $g(x) \leq 0$ a necessary conditions for optimality is that λ be non-negative (non-positive) for maximization(minimization) problems.

This result is justified by, nothing that the vector λ measures the rate of variation of f with respect to g that is $\lambda = \partial f / \partial g$

In the maximization case, as the right hand side of the constraint $g(x) \leq 0$ increases from 0 to the vector ∂g and $\lambda \geq 0$.

Similarly, for minimization, as the right hand side of the constraints increases, f cannot increases, which implies that $\lambda \leq 0$

If the constraints are equalities that is $g(x) = 0$, then λ becomes unrestricted in sign.

12.6 Keywords

The KKT necessary conditions for maximization problems are summarized as $\lambda \geq 0$

$$\nabla f(x) - \lambda \nabla g(x) = 0$$

$\lambda_i g_i(x) = 0$, i=1,2,...n

$$g(x) \leq 0$$

These conditions apply to the minimization case as well, except that λ must be non-positive.

12.7 Self Assessment Questions and Exercises

1. Determine x_1, x_2, x_3 so as to
   Maximize $Z= -x^2_1-x^2_2-x^2_3+4x_1+6x_2$
   Subject to the constraints: $x_1+x_2 \leq 2$
   $2x_1+3x_2 \leq 12$
   $x_1, x_2 \geq 0$.
2. Write the Kuhn- Tucker conditions for the following minimization problem
   Min $f(x) = x^2_1+x^2_2+x^2_3$,
   $g_1(x)= 2x_1+x_2 \leq 5$
   $g_2(x)= x_1+x_3 \leq 2$
   $g_3(x)= -x_1 \leq 1$
   $g_4(x)= -x_2 \leq -2$
   $g_5(x)= -x_3 \leq 0$
3. Maximize $z = 3.6x_1-0.4x^2_1 +1.6x_2 -0.2x^2_2$ subject to the constraints $2x_1+x_2 \leq 10$ and
   $x_1, x_2 \geq 0$.
4. Determine x_1, x_2, x_3 so as to
   Maximize $Z= -2x^2_1-x^2_2-6x^2_3+x_1+x_2$
   Subject to the constraints: $x_1+x_2 \leq 12$
   $2x_1+x_2 \leq 6$
   $x_1, x_2 \geq 0$.
5. Determine x_1 and x_2 so as to
   Maximize $Z=2x_1x_2 -2x^2_1-2x^2_2+12x_1+21x_2$
   Subject to the constraints: $x_2 \leq 8$
   $x_1+x_2 \leq 10$, $x_1, x_2 \geq 0$.

12.8 Further Readings

11. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi,2008.

UNIT XIII NON LINEAR PROGRAMMING – UNCONSTRAINTS ALGORITHMS

Structure

13.1 Introduction

This chapter deals with the concept of developable surfaces. In the plane,we find the envelope of straight lines. The extension of the notion of envelope of straight lines to envelope of planes in space leads to what are called developable surfaces. When we specialise to different planes such as osculating plane at a point on a space curve, we different developable surfaces.

13.2 Objectives

After going through this unit, you will be able to:

- Understand the types of unconstraint algorithms.
- Explain direct search methods
- Solve the problems in unconstraint algorithms

13.3 Non linear programming- Unconstraint algorithms

There are two types of method in the unconstrained problem. They are the direct search algorithm and the gradient algorithm.

Direct search method:

Direct search methods apply primarily to strictly uni-model single variable functions. The idea of direct search method is to identify the interval of uncertainty that is known to include the optimum solution point. The procedure locates the optimum by iteratively narrowing the interval of uncertainty to any desired level of accuracy.

Two closely related algorithms are presented in this section: dichotomus and golden section search methods. Both algorithms seek the maximization of a unimodal function f(x) over the interval $a \leq x \leq b$, which is known to include the optimum point x*. The two method start with $I_0=(a,b)$ representing the initial interval of uncertainty.

General step i.

Let $I_{i-1}=(x_L, x_R)$ be the current interval of uncertainty (at iteration 0, $x_L=a$ and $x_R=b$). Next, identify x_1 and x_2 in the following manner.

Dichotomus method:

$X_1 = \frac{1}{2}(x_R + x_L - \Delta)$

$X_2 = \frac{1}{2}(x_R + x_L + \Delta)$

Golden section method:

$X_1 = x_R - (\frac{\sqrt{5}-1}{2})(x_R - x_L)$

$X_2 = x_L + (\frac{\sqrt{5}-1}{2})(x_R - x_L)$

The selection of x_1 and x_2 guarantees that $x_L < x_1 < x_2 < x_R$

The next interval of uncertainty, I_i is determined in the following manner.

1. If $f(x_1)>f(x_2)$, then $x_L < x^* < x_2$. Let $x_R = x_2$ and set $I_i=(x_L, x_2)$
2. If $f(x_1)<f(x_2)$, then $x_1 < x^* < x_R$. Let $x_L = x_1$ and set $I_i=(x_1, x_R)$
3. If $f(x_1)=f(x_2)$, then $x_1 < x^* < x_2$. Let $x_L = x_1$ and set $I_i=(x_1, x_2)$

The manner in which x_1 and x_2 are determined guarantees that $I_{i+1}<I_i$, as will be shown shortly. The algorithm terminates at iteration k if $I_k \leq \Delta$, where Δ is a user-specified level of accuracy.

In the dichotomous method, the values x_1 and x_2 sit symmetrically around the mid point of the current interval of uncertainty. This means that $I_{i+1}=0.5(I_i + \Delta)$

Repeated application of the algorithm guarantees that the length of the interval of uncertainty will approach the desired accuracy, Δ.

In the golden section method, the idea is more involved. The golden section proposes is to save computations by reusing the discarded value in the immediately succeeding iteration.

Define for $0 < \alpha < 1$,

$X_1 = x_R - \alpha(x_R - x_L)$

$X_2 = x_L + \alpha(x_R - x_L)$

Then the interval of uncertainty I_i at iteration i equals (x_L, x_2) or (x_1, x_R). Consider the case $I_i= (x_L, x_2)$, which means that x_1 is included in I_i. In iteration i+1, we select x_2 equal to x_1 in iteration i, which leads to the following equation:

X_2(iteration i+1) = x_1 (iteration i)

Substitution yields

$X_L+\alpha[X_2(\text{iteration i})- X_L] = X_R-\alpha(X_R - X_L)$

Or $X_L+\alpha[X_L+\alpha(X_R - X_L)- X_L] = X_R-\alpha(X_R - X_L)$

Which is finally simplifies to $\alpha^2 + \alpha - 1 = 0$

This equation yields $\alpha = \frac{-1\pm\sqrt{5}}{2}$. Because $0 \leq \alpha \leq 1$, we select the positive root $\alpha = \frac{-1\pm\sqrt{5}}{2} \approx .681$

1. Maximize $f(x) = \begin{cases} 3x, & 0 \leq x \leq 2 \\ \frac{1}{3}(-x+20), & 2 \leq x \leq 3 \end{cases}$

The maximum value of f(x) occurs at x=2. The following table demonstrates the calculations for iteration 1 and 2 using the dichotomous and the golde section methods.

Solution:

We will assume $\Delta = 0.1$

Dichotomous method:

Iteration:1

$I_0 =(0, 3) \equiv (x_L, x_R)$

$X_1 = 0+.5(3-0-.1) =1.45$

$f(x_1)=4.35$

$X_2 = 0+.5(3-0+.1) =1.55$

$f(x_2)=4.65$

$f(x_2)> f(x_1)$

$X_L =1.45$, $I_1 =(1.45, 3)$

Iteration:2

$I_1 = (1.45, 3) \equiv (x_L, x_R)$

$X_1 = 1.45 + .5(3 - 1.45 - .1) = 2.175$

$f(x_1) = 5.942$

$X_1 = \frac{3+1.45+.1}{2} = 2.275$

$f(x_2) = 5.908$

$f(x_1) > f(x_2)$

$X_R = 2.275$, $I_2 = (1.45, 2.275)$

Golden section method:

Iteration:1

$I_0 = (0, 3) \equiv (x_L, x_R)$

$X_1 = 3 - .618(3 - 0) = 1.146$

$f(x_1) = 3.438$

$X_2 = 0 + .618(3 - 0) = 1.854$

$f(x_2) = 5.562$

$f(x_2) > f(x_1)$

$X_L = 1.146$, $I_1 = (1.146, 3)$

Iteration:2

$I_1 = (1.146, 3) \equiv (x_L, x_R)$

$X_1 = X_2$ in iteration 0=1.854

$f(x_1) = 5.562$

$X_1 = 1.146 + .618(3 - 1.146) = 2.292$

$f(x_2) = 5.903$

$f(x_2) > f(x_1)$

$X_L = 1.854$, $I_1 = (1.854, 3)$

Continuing in the same manner, the interval of uncertainty will eventually narrow down to the desired Δ-tolerance.

Gradient Method

This section develops a method for optimizing that are twice continuously differentiable. The idea is to general successive points in the direction of the gradient of the function.

The Newton-Raphson method a gradient method for solving simultaneous equations. This section presents another technique, called the **Steepest ascent** method.

Termination of the gradient method occurs at the point where the gradient vector becomes null. This is only a necessary condition for optimality. Optimality cannot be verified unless it is known a priori that f(**X**) is concave or convex.

Suppose that f(X) is maximized. Let X_0 be the initial point from which the procedure starts and define $\nabla f(X_k)$ as the gradient of f at point X_k. The idea is to determine a particular path p along which $\frac{\partial f}{\partial p}$ is maximized at a given point. This result is achieved if successive points X_k and X_{k+1} are selected such that $X_{k+1} = X_k + r_k \nabla f(X_k)$, where r_k is the optimal step size at X_k.

The step size r_k is determined such that the next point, X_{k+1}, leads to the largest improvement in f. This is equivalent to determining r= r_k that maximize the function $h(r) = f[X_k + r_k \nabla f(X_k)]$.

Because h(r) is a single variable function. The proposed procedure terminates when two successive trial points X_k and X_{k+1} are approximately equal. This is equivalent to having $r_k \nabla f(X_k) \approx 0$. Because $r_k \neq 0$, the necessary condition $\nabla f(X_k)=0$ is satisfied at X_k.

1. Consider the following problem

   Maximize $f(x_1, x_2)= 4x_1+6x_2-2x_1^2-2x_1x_2-2x_2^2$

   The exact optimum occurs at $(x^*_1, x^*_2) = (\frac{1}{3}, \frac{4}{3})$.

   We show how the problem is solved by the steepest ascent method. The gradient of f is given as

   $\nabla f(X) = (4-4x_1-2x_2, 6-2x_1-4x_2)$

   The quadratic nature of the function dictates that the gradients at any two successive points are orthogonal.

   Suppose that we start at the initial point $X_0 =(1,1)$.

   Iteration:1

   $\nabla f(X_0) =(-2,0)$

   The next point X_1 is obtained by considering X= (1,1)+r(-2,0)= (1-2r,1)

   Thus, $h(r) = f(1-2r, 1) =-2(1-2r)^2+2(1-2r)+4$

   The optimal step size is obtained using the classical necessary conditions. The maximum value of h(r) is $r_1=1/4$, which yields the next solution point as $X_1 =(\frac{1}{2}, 1)$.

   Iteration:2

   $\nabla f(X_1) =(0,1)$

   $X=(\frac{1}{2}, 1)+r(0,1) = (\frac{1}{2}, 1+r)$

   $h(r)= -2(1+r)^2 +5(1+r)+\frac{3}{2}$

   This gives $r_2 =\frac{1}{4}$ and $x_2 =(\frac{1}{2}, \frac{5}{4})$.

   Iteration:3

   $\nabla f(X_2) =(0, \frac{1}{4})$

   $X=(\frac{1}{2}, \frac{5}{4})+r(-\frac{1}{2},0) = (\frac{1-r}{2}, \frac{5}{4})$

   $h(r)= -\frac{1}{2}(1-r)^2 +\frac{3}{4}(1-r)+\frac{35}{8}$

   Hence, $r_3 =\frac{1}{4}$ and $x_3 =(\frac{3}{8}, \frac{5}{4})$.

   Iteration:4

   $\nabla f(X_3) =(0, \frac{1}{4})$

   $X=(\frac{3}{8}, \frac{5}{4})-r(0,\frac{1}{4}) = (\frac{3}{8}, \frac{5+r}{4})$

   $h(r)= -\frac{1}{8}(5+r)^2 +\frac{21}{16}(5+r)+\frac{39}{32}$

   Thus, $r_4 =\frac{1}{4}$ and $x_4 =(\frac{3}{8}, \frac{21}{16})$.

   Iteration:5

   $\nabla f(X_4) =(-\frac{1}{8}, 0)$

   $X=(\frac{3}{8}, \frac{21}{16})+r(-\frac{1}{8},0) = (\frac{3-r}{8}, \frac{21}{16})$

   $h(r)= -\frac{1}{32}(3-r)^2 +\frac{11}{64}(3-r)+\frac{567}{128}$

   Hence, $r_5 =\frac{1}{4}$ and $x_5 =(\frac{11}{32}, \frac{21}{16})$.

   Iteration:6

   $\nabla f(X_5) =(0, \frac{1}{16})$

Because $\nabla f(X_5)\approx 0$, the process can be terminated at this point. The approximate maximum point is given by X_5=(.3438, 1.3125). The exact optimum X* =(.3333, 1.3333).

13.4 Check your progress

- State unconstraint algorithms
- Define unimoidal function
- Define interval of uncertainty

13.5 Summary

- The next interval of uncertainty, I_i is determined in the following manner.If $f(x_1)>f(x_2)$, then $x_L < x^* < x_2$.
  Let $x_R = x_2$ and set $I_i=(x_L, x_2)$
  If $f(x_1)<f(x_2)$, then $x_1 < x^* < x_R$. Let $x_L = x_1$ and set $I_i=(x_1, x_R)$
  If $f(x_1)=f(x_2)$, then $x_1 < x^* < x_2$. Let $x_L = x_1$ and set $I_i=(x_1, x_2)$
- The Newton-Raphson method a gradient method for solving simultaneous equations
- Termination of the gradient method occurs at the point where the gradient vector becomes null. This is only a necessary condition for optimality. Optimality cannot be verified unless it is known a priori that f(**X**) is concave or convex.

13.6 Keywords

Dichotomus method:

$X_1 = \frac{1}{2}(x_R + x_L - \Delta)$

$X_2 = \frac{1}{2}(x_R + x_L + \Delta)$

Golden section method:

$X_1 = x_R - (\frac{\sqrt{5}-1}{2})(x_R - x_L)$

$X_2 = x_L + (\frac{\sqrt{5}-1}{2})(x_R - x_L)$

13.7 Self Assessment Questions and Exercises

1. Find the maximum of each of the following functions by dichotomous search. Assume that
   Δ =.05.
   a) $f(x)= \frac{1}{|(x-3)^3|}$, $2\leq x \leq 4$
   b) $f(x)= x \cos x$, $0\leq x \leq \pi$
   c) $f(x)= x \sin\pi x$, $1.5 \leq x \leq 2.5$
   d) $f(x)= -(x-3)^2$, $2\leq x \leq 4$
2. Show that, in general, the Newton-Raphson method when applied to a strictly concave quadratic function will converge in exactly one step. Apply the method to the maximization of
   $f(X) = 4x_1+6x_2-2x_1^2-2x_1x_2 - 2x_2^2$
3. Carry out at most five iteration for each of the following problems using the method of steepest descent (ascent). Assume that $X^0 = 0$ in each case.
   a) Min f(x) = min $f(X) = (x_2 - x_1^2)^2+(1-x_1)$
   b) Max $f(X) = cX+X^T AX$

NOTES

Where c= (1,3,5)

$$A = \begin{matrix} -5 & -3 & -1/2 \\ -3 & -2 & 0 \\ -1/2 & 0 & -1/2 \end{matrix}$$

Min f(X) = $x_1 - x_2 + x^2{}_1 - x_1 x_2$

13.8 Further Readings

11. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi,2008.

UNIT XIV
SEPARABLE AND QUADRATIC PROGRAMMIING

Structure

13.1 Introduction

This chapter introduces the concept of separable and quadratic programming. Also these methods are explained with suitable examples.

13.2 Objectives

After going through this unit, you will be able to:

- Define seperable function
- Understand the aim of separable and quadratic programming.
- Solve the problems in separable and quadratic programming

13.3 Separable and Quadratic programming

Constrained Algorithm:

The general constrained nonlinear programming problem is defined as

Maximize(or minimize)z = f(X)

Subject to: $g(X) \leq 0$

The non negativity condition, $X \geq 0$, are part of the constraints. Also at least one of the functions f(X) and g(X) is non linear and all the function are continuously differentiable.

The most general result applicable to the problem is the KKT conditions. The KKT condition are only necessary for realizing optimality. A number of algorithms that may be classified generally as indirect and direct method. Indirect methods solve the non linear problem by dealing with one or more linear programs derived from the original programs. Direct methods deals with the original problem.

Separable Programming

A function $f(x_1, x_2,.......,x_n)$ is separable if it can be expressed as the sum of n single variable function $f_1(x_1), f_2(x_2),......, f_n(x_n)$ that is ,

$$f(x_1, x_2,.......,x_n)= f_1(x_1)+ f_2(x_2)+......+ f_n(x_n)$$

for example, the linear function

$$h(x_1, x_2,.......,x_n)= a_1(x_1)+ a_2(x_2)+......+ a_n(x_n)$$

a separable (the parameters $a_i.i = 1,2,.....,n$ are constant). Conversely the function

$$h(x_1, x_2,.......,x_n)= x_1^2+ x_1 \sin(x_2 + x_3)+......+ x_2 e^{x3}$$ is not separable.

Some nonlinear functions are not directly separable but can be made so by appropriate substitutions. The single variable function f(X) can be approximated by a piecewise linear function using mixed integer programming. Suppose that f(X) is to be approximated over an interval[a,b]. Define a_k, k=1,2,....,k, as the kth breakpoint on the x-axis such that $a_1 < a_2 < ..... < a_k$. The points a_1 and a_k coincide with end points a and b of the designated interval. thus, f(x) is approximated as follows:

$$f(x) \approx \sum_{k=1}^{K} f(a_k).w_k$$

$$x = \sum_{k=1}^{K} a_k .w_k$$

where w_k is a non negative weight associated with the kth breakpoint such that,

$$\sum_{k=1}^{K} w_k = 1,\ w_k \geq 0,\ k=1,2,.....k$$

Mixed integer programming ensures the validity of the approximation by imposing two conditions:

1. At most two w_k are positive.
2. If w_k is positive, then only an adjacent w_{k+1} or w_{k-1} can assume a positive value. To show how these conditions are satisfied,

Consider the separable problem

$$\text{Maximize(or minimize)}z = \sum_{j=1}^{n} f_j(X_j)$$

$$\text{Subject to } \sum_{j=1}^{n} g_{ji}(X_j) \leq b_i,\ i= 1,2,.....,m$$

This problem can be approximated by a mixed integer program as follows, let[1]

`

a_{kj} = breakpoint k for variable x_j

w_{kj} = weight with breakpoint k of variable x_j } k=1,2,.....,k_j , j=1,2,.....,n

then the equivalent mixed problem is,

$$\text{Maximize(or minimize)}z = \sum_{j=1}^{n} \sum_{k=1}^{k_j} f_j (w_{kj})(a_{kj})$$

$$\text{Subject to } \sum_{j=1}^{n} \sum_{k=1}^{k_j} g_j (w_{kj})(a_{kj}) \leq b_i,\ i=1,2,......,m$$

$$\sum_{k=1}^{k_j} (y_{kj}) =1, \quad j=1,2,.....,n$$

$$\sum_{k=1}^{k_j} (w_{kj}) =1, \quad j=1,2,.....,n$$

The variable for the approximating problem are w_{jk} and y_{jk}.

1. Consider the problem

$$\text{Maximize } z = x_1 + x_2^2$$

Subject to

$$3 x_1 + 2x_2^2 \leq 9$$

$$x_1, x_2 \geq 0$$

The exact optimum solution to this problem, obtained by AMPL or solver is x_1=0, x_2=2.2132 and z*=20.25. To show how the approximating method is used, consider the separable functions,

$f_1(x_1) = x_1$

$f_2(x_2) = x_2^4$

$g_1(x_1) = 3x_1$

$g_2(x_2) = 2x_2^2$

The function $f_1(x_1)$ and $g_1(x_1)$ remain the same because they are already linear. In this case, x_1 is treated as one of the variables. Considering $f_2(x_2)$ and $g_2(x_2)$, we assume four breakpoints: a_{2k} =0,1,2 and 3 for k=1,2,3 and 4 respectively. Because the value of x_2 cannot exceed 3, it follows that

K	a_{2k}	$f_2(a_{2k})=a_{2k}^4$	$g_2(a_{2k})=2a_{2k}^2$
1	0	0	0
2	1	1	2
3	2	16	8
4	3	81	18

NOTES

This yields $f_2(x_2) \approx w_{21} f_2(a_{21}) + w_{22} f_2(a_{22}) + w_{23} f_2(a_{23}) + w_{24} f_2(a_{24})$

$\approx 0w_{21}+1w_{22}+16w_{23}+81w_{24}$ =

$w_{22}+16w_{23}+81w_{24}$

Similarly,

$g_2(x_2) \approx 2w_{22}+8w_{23}+18w_{24}$

The approximating problem thus becomes

Maximize $z= x_1+w22+16w_{23}+81w_{24}$

Subject to

$3 x_1+2w_{22}+8w_{23}+18w_{24} \leq 9$

$w_{21}+w_{22}+w_{23}+w_{24}=1$

$x_1 \geq 0,\ w_{2k} \geq 0, k = 1,2,3,4$

the values of w_{2k}, k=1,2,3,4, must satisfy the restricted basis condition.

The initial simplex tableau is given by

Basic	X_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}	solution
Z	-1	-1	-16	-81	0	0	0
s_1	3	2	8	18	1	0	9
w_{21}	0	1	1	1	0	1	1

The variable $s_1(\geq 0)$ is a slack.

From the z-row coefficient, w_{24} is the entering variable. Because w_{21} is currently basic and positive, the restricted basis condition dicates that it must leave before w_{24} can enter the solution. By the feasibility condition, s_1 must be the leaving variable, which means that w_{24} cannot enter the solution. The next best entering variable, w_{23}, require w_{21} to leave the basic solution, a condition that happens to be satisfied by the feasibility condition.

The new tableau thus becomes

Basic	X_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}	solution
Z	-1	15	0	-65	0	16	16
s_1	3	-6	0	10	1	-8	1
w_{23}	0	1	1	1	0	1	1

Now, w_{24} is the entering variable, which is admissible because w_{23} is positive. The simplex method shows that s_1 will leave. Thus,

Basic	X_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}	solution
Z	$\frac{37}{2}$	-24	0	0	$\frac{37}{2}$	-36	$22\frac{1}{2}$
W_{24}	3/10	6/10	0	1	1/10	8/10	1/10
w_{23}	3/10	16/10	1	0	-1/10	18/10	9/10

The tableau shows that w_{21} and w_{22} are candidates for the entering variables. Because w_{21} is not adjacent to basic w_{23} 0r w_{24}, it cannot enter. Similarly, w_{22} cannot enter because w_{24} cannot leave. The last tableau thus is the best restricted basis solution for the approximate problem.

The optimum solution to the original problem is x_1=0

$X_{23} \approx 2w_{23}+3w_{24} = 2(\frac{9}{10})+3(\frac{1}{10}) = 2.1$

$Z = 0+2.1^4 = 1.45$

The value x_2 = 2.1 approximately equals the true optimum value (=2.12132).

Separable Convex Programming:

A special case of separable programming occurs when $g_{ij}(x_j)$ is convex for all i and j, which ensures a convex solution space. Additionally, if $f_j(x_j)$ is convex (minimization) or concave (maximization) for all j, then the problem has a global optimum. Under such conditions, the following simplified approximation can be used.

Consider a minimization problem and $f_j(x_j)$. The breakpoints of the function $f_j(x_j)$ are $x_j=a_{jk}$, k=0,1,..K_j. Let x_{jk} define the increment of the variable x_j in the range ($a_{j,k-1}$, a_{jk}), let r_{jk} be the corresponding rate of change in the same range. Then

$$f_j(x_j) \approx \sum_{k=1}^{k_j} r_{jk} x_{jk} + f_j(a_{j0})$$

$$x_j = \sum_{k=1}^{k_j} x_{jk}$$

$$0 \leq x_{jk} \leq a_{jk} - a_j, \ k=1,2,...k_j$$

The fact that $f_j(x_j)$ is convex ensures that $r_{j1} < r_{j2} < .... < r_{jk}$. This means that in the minimization problem, for p<q, the variable x_{jp} is more attractive than x_{jq}. Consequently, x_{jp} will always reach its maximum limit before x_{jq} can assume a positive value.

The convex constraint function $g_{ij}(x_j)$ are approximated in essentially the same way. Let r_{ijk} be the slope of the kth line segment corresponding to $g_{ij}(x_j)$. It follows that

$$g_{ij}(x_j) \approx \sum_{k=1}^{k_j} r_{ijk} x_{jk} + g_{ij}(a_{j0})$$

The complete problem is thus given by

$$\text{Minimize } z=\sum_{j=1}^{n}\left(\sum_{k=1}^{k_j} r_{jk} x_{jk} + f_j(a_{j0})\right)$$

Subject to

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{k_j} r_{ijk} x_{jk} + g_{ij}(a_{j0})\right) \leq b_i$$

$$0 \leq x_{jk} \leq a_{jk} - a_j, \ k=1,2,...k_j, \ j=1,2,..,n$$

$$\text{Where } r_{jk} = \frac{f_j(a_{jk}) - f_j(a_{j,k-1})}{a_{jk} - a_{j,k-1}}$$

$$r_{ijk} = \frac{g_{ij}(a_{jk}) - g_{ij}(a_{j,k-1})}{a_{jk} - a_{j,k-1}}$$

The maximization problem is treated essentially the same way.

In this case, $r_{j1} > r_{j2} >,....,> r_{jk}$ which means, for p<q, the variable x_{jp} will always reach its maximum limit before x_{jq} can assume a positive value.

The new problem can be solved by the simplex method with upper bounded variables. The restricted basis concept is not needed because the convexity of the functions guarantees correct selection of basic variables.

1. Consider the problem Maximize $z = x_1 - x_2$

   Subject to $3x_1^4 + x_2 \leq 243$

   $X_1 + 2x_2^2 \leq 32$

   $X_1 \geq 2.1$

   $X_2 \geq 3.5$

The separable functions of this problem are

$f_1(x_1) = x_1$, $f_2(x_2) = -x_2$

$g_{11}(x_1) = 3x_1^4$, $g_{12}(x_2) = x_2$

$g_{21}(x_1) = x_1$, $g_{22}(x_2) = 2x_2^2$

These functions satisfy the convexity condition required for the minimization problems. The function $f_1(x_1)$, $f_2(x_2)$, $g_{12}(x_2)$ and $g_{21}(x_1)$ are already linear and need not be approximated.

K	a_{1k}	$g_{11}(a_{1k})=3a_{1k}^4$	r_{11k}	X_{1k}
0	0	0	-	-
1	1	3	3	X_{11}
2	2	48	45	X_{12}
3	3	243	195	X_{13}

The ranges of the variables x_1 and x_2 are $0\le x_1\le 3$ and $0\le x_2\le 4$. Let $K_1=3$ and $K_2=4$. The slopes corresponding to the separable function are determined as follows.

For j = 1,

For j = 2,

K	a_{2k}	$g_{22}(a_{2k})=2a_{2k}^2$	r_{22k}	X_{2k}
0	0	0	-	-
1	1	2	2	X_{21}
2	2	8	6	X_{22}
3	3	18	10	X_{23}
4	4	32	14	X_{24}

The complete problem ten becomes

Maximize $z = x_1 - x_2$

Subject to

$3x_{11}+45x_{12}+195x_{13}+x_2\le 243$ ....(1)

$x_1+2x_{21}+6x_{22}+10x_{23}+14x_{24}\le 32$ ....(2)

$X_1\ge 2.1$ ....(3)

$X_2\ge 3.5$ ....(4)

$x_{11}+x_{12}+x_{13}-x_1=0$ ....(5)

$x_{21}+x_{22}+x_{23}+x_{24}-x_2=0$ ....(6)

$0\le x_{1k}\le 1$, k=1,2,3 ....(7)

$0\le x_{2k}\le 1$, k=1,2,3,4 ....(8)

$x_1, x_2\ge 0$

Constraints 5 and 6 are needed to maintain the relationship between the original and new variables.

The optimum solution is

Z= -.52, $x_1=2.98$, $x_2=3.5$, $x_{11}=x_{12}=1$, $x_{13}=.98$, $x_{21}=x_{22}=x_{23}=1$, $x_{24}=.5$

Quadratic programming:

A quadratic programming model is defined as

Maximize $z=CX+X^TDX$

Subject to $AX\le b$, $X\ge 0$

Where $X=(x_1, x_2, \ldots, x_n)^T$

$C=(c_1, c_2, \ldots, c_n)$

$B=(b_1, b_2, \ldots, b_m)^T$

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$D=\begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}$$

The function X^TDX defines a quadratic form. The matrix D is assumed symmetric and negative definite. This means that z is strictly concave. The constraints are linear, which guarantees a convex solution space.

The solution to this problem is based on the KKT necessary conditions. Because z is strictly concave and the solution space is a convex set, these condition are also sufficient for a global optimum.

The quadratic programming problem will be treated for the maximization case. Conversion to minimization is straightforward. The problem may be written a

Maximize z= CX+X^TDX

Subject to G(X) = $\binom{A}{-I}$X - $\binom{b}{0}$ ≤ 0

Let λ =$(\lambda_1, \lambda_2, \ldots, \lambda_m)^T$

U=$(\mu_1, \mu_2, \ldots, \mu_n)^T$

Be the Lagrange multipliers corresponding to constraints AX-b ≤ 0 and –X ≤ 0, respectively.

Application of the KKT conditions yields

λ≥0, U≥0

$$\nabla z - (\lambda^T, U^T)\nabla G(X) = 0$$

$\lambda_i(b_i - \sum_{j=1}^{n} a_{ij}\, x_j)$ -0, i=1,2,...,m

$\mu_j x_j$ = 0, j=1,2,..,n

$$AX \le b$$

-X≤ 0

Now ∇z =C+2X^TD

∇G(X) = $\binom{A}{-I}$

Let S=b-AX≥ 0 be the slack variables of the constraints. The conditions reduce to

-2X^TD+$\lambda^T A$-U^T=C

AX+S=b

$\mu_j x_j$ = 0, $\lambda_i S_i$ for all i and j

$$\lambda, U, X, S \ge 0$$

Because D^T=D, the transpose of the first set of equation can be written as

-2DX+A^T**λ**-U=C^T

Hence, the necessary condition may be combined as

$$\begin{pmatrix} -2D & A^T - I & 0 \\ A & 0 & 0 & I \end{pmatrix}\begin{pmatrix} X \\ \lambda \\ U \\ S \end{pmatrix}=\begin{pmatrix} C^T \\ b \end{pmatrix}$$

$\mu_j x_j$ = 0 = $\lambda_i S_i$ for all i and j

$$\lambda, U, X, S \ge 0$$

Except the condition $\mu_j x_j$ = 0 = $\lambda_i S_i$, the remaining equations are linear functions in X,**λ**,U and S. Thus, the problem is equivalent to solving a set of linear equation with the additional conditions $\mu_j x_j$ = 0 = $\lambda_i S_i$. Because z is strictly concave and the solution space is convex, the feasible solution satisfying all these conditions must yields a unique optimum solution.

The solution of the system is obtained by using phase I of the two-phase method. The only restriction is to satisfy the condition $\mu_j x_j$ = 0 = $\lambda_i S_i$. This means that $\lambda_i\, and\, S_i$ cannot be positive simultaneously, and neither can $\mu_j\, and\, x_j$.

1. Consider the problem

   Maximize z =4$x_1 + 6x_2$-2x_1^2-2x_1x_2-2x_2^2

   Subject to

   $x_1 + 2x_2$≤2

$x_1, x_2 \ge 0$.

This problem can be put in matrix form as follows:

Maximize z = (4, 6)$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$+$(x_1, x_2)$$\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Subject to

(1, 2) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$≤ 2

$x_1, x_2 \geq 0.$

The KKT conditions are given as

$\begin{pmatrix} 4 & 2 & 1 & -1 & 0 & 0 \\ 2 & 4 & 2 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$, $\mu_1 x_1$ = $\mu_2 x_2$ = $\lambda_1 S_1$ =0

The initial tableau for phase 1 is obtained by introducing the artificial variable R_1 and R_2 and updating objective row. Thus

Basic	x_1	x_2	λ_1	μ_1	μ_2	R_1	R_2	S_1	Solution
R	6	6	3	-1	-1	0	0	0	10
R_1	4	2	1	-1	0	1	0	0	4
R_2	2	4	2	0	-1	0	1	0	6
S_1	1	2	0	0	0	0	0	1	2

Iteration:1

Because μ_1=0, the most promising entering variables x_1 can be made basic with

R_1 as the leaving variable. This yields the following tableau:

Basic	x_1	x_2	λ_1	μ_1	μ_2	R_1	R_2	S_1	Solution
R	0	3	3/2	1/2	-1	-3/2	0	0	4
x_1	1	½	1/4	-1/4	0	¼	0	0	1
R_2	0	3	3/2	1/2	-1	½	1	0	4
S_1	0	3/2	-1/4	1/4	0	-1/4	0	1	1

Iteration:2

The most promising variable x_2 can be made basic because μ_2=0. This gives

Basic	x_1	x_2	λ_1	μ_1	μ_2	R_1	R_2	S_1	Solution
R	0	0	2	0	-1	-1	0	-2	2
x_1	1	0	1/3	-1/3	0	1/3	0	-1/3	2/3
R_1	0	0	2	0	-1	0	1	-2	2
x_1	0	1	-1/6	1/6	0	-1/6	0	2/3	2/3

Iteration:3

Because S_1=0, λ_1 can be introduced into the solution. This yields

Basic	x_1	x_2	λ_1	μ_1	μ_2	R_1	R_2	S_1	Solution
R	0	0	0	0	0	-1	-1	0	0
x_1	1	0	0	-1/3	1/6	1/3	-1/6	0	1/3
λ_1	0	0	1	0	-1/2	0	1/2	-1	1
x_2	0	1	0	1/6	-1/12	-1/6	1/12	½	5/6

The last tableau gives the optimal solution for phase I. Because r=0, the solution, x_1^*=1/3, x_2^*= 5/6, is feasible. The optimal value z, computed from the original problem is 4.16.

14.4 Check your progress

- Define separable function
- State non negativity conditions in separable programming
- Define quadratic programming

14.5 Summary

- The general constrained nonlinear programming problem is defined as

Maximize(or minimize)z = f(X)

Subject to: g(X)≤ 0

The non negativity condition, X ≥ 0, are part of the constraints. Also at least one of the functions f(X) and g(X) is non linear and all the function are continuously differentiable

A function $f(x_1, x_2,.......,x_n)$ is separable if it can be expressed as the sum of n single variable function $f_1(x_1)$, $f_2(x_2)$,......, $f_n(x_n)$ that is ,

$f(x_1, x_2,.......,x_n)= f_1(x_1)+ f_2(x_2)+......+ f_n(x_n)$

14.6Keywords

Separable Function: A function $f(x_1, x_2,.......,x_n)$ is separable if it can be expressed as the sum of n single variable function $f_1(x_1)$, $f_2(x_2)$,......, $f_n(x_n)$ that is ,

$f(x_1, x_2,.......,x_n)= f_1(x_1)+ f_2(x_2)+......+ f_n(x_n)$

Mixed integer programming ensures the validity of the approximation by imposing two **conditions**:

1. At most two w_k are positive.
2. If w_k is positive, then only an adjacent w_{k+1} or w_{k-1} can assume a positive value..

14.7 Self Assessment Questions and Exercises

1. Use separable programming algorithm to the non-linear programming problem
   Max $z= x_1+x_2^2$
   Subject to the constraints $3x^2_1+2x^2_2 \leq 9$, $x_1 \geq 0$, $x_1 \geq 0$.
2. Show how the following problem can be made separable.
   Max $z= x_1x_2+x_3+x_1x_3$
   Subject to $x_1x_2+x_3+x_1x_3\leq 10$
   x_1, $x_2x_3\geq0$.
3. Show how the following problem can be made separable.
   Max $z= e^{x_1+x^2_2}+(x_3-2)^2$
   Subject to $x_1+x_2+x_3\leq 6$
   x_1, $x_2x_3\geq0$.
4. Show how the following problem can be made separable.
   Max $z= e^{x_1x2}+x_2^2x_3+ x_4$
   Subject to $x_1+x_2x_3+x_3\leq 10$
   x_1, $x_2x_3\geq0$
   x_4 unrestricted in sign.
5. Show that in separable convex programming, it is never optimal to have $x_{ki}> 0$ when $x_{k-1,I}$ is not at its upper bound.
6. Solve as a separable convex programming problem.
   Minimize $z= x_1^4+x_2+x_3^2$
   Subject to $x_1^2+x_2+x_3^2\leq4$
   $|x_1 + x_2|\leq 0$
   x_2, $x_3\geq0$
   x_2 unrestricted in sign.

7. Use Wolf's method to solve the Quadratic programming problem
   Maximize $z=2x_1+3x_2-2x_2^2$
   Subject to $x_1+4x_2\leq 4$, $x_1+x_2\leq 2$, $x_1, x_2\geq 0$.

14.8 Further Readings

1. Operations Research, H.A.Taha, Eighth edition, Prentice Hall, New Delhi,2008.