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(A State University Established by the Government of Tamilnadu)

KARAIKUDI-630 004

DIRECTORATE OF DISTANCE EDUCATION

B. Sc. (Mathematics)

III-SEMESTER

11334

MECHANICS

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MECHANICS

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INTRODUCTION

Mechanics is a branch of the physical sciences concerned with the state of rest or motion of bodies that are subjected to the action of forces. The study of mechanics involves many more subject areas. However, initial study is usually split into two areas; **Statics and Dynamics**.

Statics is concerned with bodies that are either at rest or move with a constant speed in a fixed direction. It utilizes principles of physics and calculus. It is fundamental in many different branches of engineering, from mechanical to civil engineering, and the principles of equilibrium, moment of inertia, and center of gravity will be revisited in more advanced fields. It is because an understanding of these topics is so crucial that statics does not cover a wide range of topics. Every problem will deal with some combination of two equations: the net forces being equal to zero, and/or net moments being equal to zero.

Dynamics is the study of bodies in motion. Dynamics is concerned with describing motion and explaining its causes. The general field of dynamics consists of two major areas: kinematics and kinetics. Each of these areas can be further divided to describe and explain linear, angular, or general motion of bodies. The fundamental concepts in dynamics are space (relative position or displacement), time, mass, and force. Other important concepts include velocity, acceleration, torque, moment, work, energy, power, impulse, and momentum. The broad definitions of basic terms and concepts in dynamics will be introduced in this chapter.

Basic Concepts

Time is the measure of a succession of events and is a basic quantity in dynamics. Time is not involved in the analysis of statics problems. Time is a scalar quantity.

Length is needed to locate the position of a point in space and describes the size of a physical system. Once a standard unit of length has been defined, it is possible to define distances and geometric properties of a body as a multiple of the unit of length. Length is a scalar quantity.

Volume is a measurement of the physical size of an object. It refers to how much space an object takes up. Volume is a scalar quantity.

Mass is a different measurement of the size of an object. The mass, measured in kilograms, depends only on the amount of matter forming the body. Mass is a scalar quantity.

Density is related to mass and volume. It is defined as the mass per unit volume. This means that an object that has a large mass but a small volume will have a large density. Density is a scalar quantity.

Equilibrium is a number of forces act on a body and keep it at rest, the forces are said to be a in equilibrium

Speed is a measure of how quickly a body is moving. It is defined as distance travelled per unit time. Speed is a scalar quantity. Forces are influences on a body or system which, acting alone would cause the motion of that body or system to change. A system or body at rest and then subjected to a force will start to move. To work with forces we need to know the magnitude (size), direction and the point of application of the force. Forces are vector quantities.

Displacement is a measure of distance in a particular direction. Displacement is a vector quantity. Velocity is the rate of change of displacement with respect to time. Velocity is a vector quantity.

Acceleration is the rate of change of velocity with respect to time. Acceleration is a vector Quantity.

Momentum is defined as the product of an object's mass and its velocity. This is a very important quantity in mechanics. It arises in many problems particularly those involving collisions. Momentum is a vector quantity.

BLOCK I
LAW OF FORCES AND RESULTANT OF FORCES

UNIT- I LAW OF FORCES

STRUCTURE

1.0 Introduction

1.1 Objectives

1.2 Resultant And Components: Definition

1.3 Simple Cases of Finding the Resultant

1.4Parallelogram of Forces: Theorem

1.5Analytical Expression for the Resultant of Two Forces

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1.0 INTRODUCTION

In this chapter, we will discuss about Forces acting at a point. First, we will know definition of Forces. After that we will discuss about which kind of forces acting at a point. We may define force as any cause which produces or tends to produce a change in the existing state of rest of a body or of its uniform motion in a straight line. A force will be completely known when we know (i) its magnitude (ii) its direction and (iii) its point of application (i.e.) the point of the body at which the force acts. Since a straight line has both magnitude and direction, a force can be conveniently represented by a straight line through the point of application. Such a straight line representing a force is called a vector. The direction of the force is indicated by the order of the letters (i.e.) AB (read as a vector AB) represents a force acting from A to B and BA represents a force acting from B to A.

Very often we represent a force acting at O in the direction OA by a straight line, say BC, parallel to OA and of suitable length. In such cases it must be understood that BC represents the force only in magnitude and direction but is not its line of action.

The force F at A along AB and the force F at B along BA are equal and opposite. Therefore, they are in equilibrium and may be removed. Thus we are left with a force F at B acting along BX and its effect is the same as the original force F at A. This is the principle of transmissibility of a force.

It is clear that the point of application of a force acting on a rigid body can be taken to be anywhere on its line of action. Thus when a force acts on a rigid body, it is not necessary to known its point of application. It is sufficient if we know its magnitude, direction and line of action.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what forces is
- Explain about Resultant of two forces
- Discuss forces and resultant of two force examples

1.2 RESULTANT AND COMPONENTS: DEFINITION

If two or more forces F_1, F_2, F_3,....... etc. act on a rigid body and if a single force R can be found whose effect on the body is the same as that of all the forces F_1, F_2, F_3,.......... etc. put together, then the single force R is called the resultant of the forces F_1, F_2, F_3,....... etc. and the forces F_1, F_2, F_3,....... etc. are called the components of the force R.

1.3 SIMPLE CASES OF FINDING THE RESULTANT

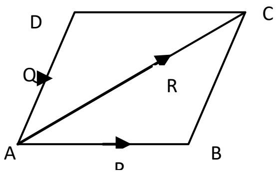
If forces P and Q act in the same direction simultaneously on a particle, the resultant is clearly equal to a forces P + Q acting in the same direction on it. If howeverP and Q act in opposite directions, their resultant is clearly equal to $P \sim Q$ and acts in direction of the greater force.

When two forces acting at a point are in different directions (i.e.) are inclined to each other, their resultant can be found with the help of a fundamental theorem in statics known as the law of the parallelogram of Forces.

1.4 PARALLELOGRAM OF FORCES: THEOREM

If two forces acting at a point be represented in magnitude and direction, by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through point.

Formal proofs of this law have been given by Bernoulli, D' Alembert and Duchayla. The law can be verified experimentally. It is assumed here and taken as the fundamental principle of statics.



Thus if the two forces P and Q acting at A are represented in magnitude and direction by the straight lines AB and AD and if the parallelogram DAB be completed, then the diagonal AC will represent in magnitude and direction the resultant of P and Q. In the language of vectors, the above law can be put as $\overline{AB} + \overline{AD} = \overline{AC}$.

1.5 ANALYTICAL EXPRESSION FOR THE RESULTANT OF TWO FORCES

Let the two forces P and Q acting at A be represented by AB and AD and let the angle between them be α

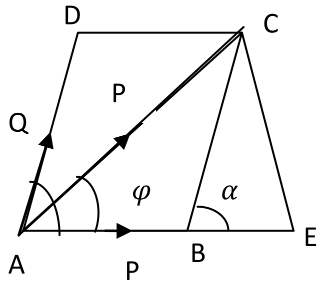
i.e. $\angle BAD = \alpha$.

Complete the parallelogram BAD. Then the diagonal AC will represent the resultant.

Let R be the magnitude of the resultant and let it make an angle φ with P i.e. $\angle CAB = \varphi$

Draw CE $\perp$ to AB. BC = AD = Q

From the right angled $\Delta\, CBE$,



$Sin\ \angle CBE = \frac{CE}{BC}$ i.e. $\sin\alpha = \frac{CE}{Q}$

$\therefore$ CE = Q sin α... ... ... (i)

$\cos\alpha = \frac{BE}{BC} = \frac{BE}{Q}$

$\therefore$ BE = Q cos α... ... ... (ii)

Now $R^2 = AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2$

$= (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$

$= P^2 + 2PQ\cos\alpha + Q^2$

NOTES

$$R^2 = \sqrt{(P^2 + 2PQ\cos\alpha + Q^2)} \text{ .... (1)}$$

Also $\tan\varphi = \frac{ce}{AE} = \frac{Q\sin\alpha}{P+Q\cos\alpha}$ ....(2)

(1) gives the magnitude and (2) the direction of the resultant in terms of P, Q and α.

Corollary 1.1:

If the forces P and Q are at right angles to each other, then $\alpha = 90°$; $\cos\alpha = cos90° = 0$ and

$\sin\alpha = sin90° = 1$

The above results become simpler and we have

(2) R = $\sqrt{P^2 + Q^2}$ and $\tan\varphi = \frac{Q}{P}$

(3) These results may be easily inferred, since the parallelogram becomes a rectangle.

Corollary 1.2:

If the forces are equal (i.e.) Q = P, then

$$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1+\cos\alpha)}$$

$$= \sqrt{2P^2 . 2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$$

and $\tan\varphi = \frac{P\sin\alpha}{P+P\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}}$$

$$= \tan\frac{\alpha}{2}$$

i.e. $\varphi = \frac{\alpha}{2}$

Thus the resultant of two equal forces P, P at an angle α is 2 P cos $\frac{\alpha}{2}$ in a direction bisecting the angle between them.

This fact (that $\varphi = \frac{\alpha}{2}$) is obvious otherwise, as the parallelogram becomes a rhombus.

Corollary 1.3:

Let the magnitudes P and Q of two forces acting at an angle α be given.

Then their resultant R is greatest when $\cos\alpha$ is greatest.

i.e. when $\cos\alpha$ =1 or $\alpha = 0°$

In this case, the forces act along the same line in the same direction and R = P + Q.

The least value of R occurs when cos α is least.

i.e. when cosα = -1 or $\alpha = 180°$

In this case, the forces act along the same line but in opposite direction and

R = $P\sim$ Q.

1.6 WORKED EXAMPLES

Example 1.1: The resultant of two forces P, Q acting at a certain angle is X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y. Prove that

$$P = (X^2 + QR)^{\frac{1}{2}} = \frac{QR(Q+R)}{Q^2+R^2-Y^2}$$

Prove also that, if P + Q + R = 0, Y = X.

Let P and Q act at an angle α

From the given data, we have the following results:

$X^2 = P^2 + Q^2 + 2PQ\cos\alpha$ ... ...(1)

$X^2 = P^2 + R^2 + 2PR\cos\alpha$ ... ...(2)

and $Y^2 = Q^2 + R^2 + 2QR\cos\alpha$ ... ...(3)

(1) – (2) gives $0 = Q^2 – R^2 + 2P\cos\alpha(Q - R)$

i.e. $0 = (Q - R)(Q + R + 2P\cos\alpha)$

But $Q \neq R$ and so Q – R is $\neq 0$

$\therefore Q + R + 2P\cos\alpha = 0$

or $\cos\alpha = -\frac{Q+R}{2P}$ ... ...(4)

Substituting (4) in (1), we have

$X^2 = P^2 + Q^2 + 2PQ.-(\frac{Q+R}{2P}) = P^2 + Q^2 – Q^2 – QR$

or $P^2 = X^2 + QR$

i.e. $P = (X^2 + QR)^{1/2}$

Substituting (4) in (3), we have

$Y^2 = Q^2 + R^2 + 2QR.-(\frac{Q+R}{2P})$

$= Q^2 + R^2 - \frac{QR(Q+R)}{p}$

NOTES

$\therefore \frac{QR(Q+R)}{p} = Q^2 + R^2 - Y^2$

or $P = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$

If P + Q + R = 0, then Q + R = -P.

$\therefore$ From (4), $\cos\alpha = -\frac{Q+R}{2P} = \frac{P}{2P} = \frac{1}{2}$

Putting $\cos\alpha = \frac{1}{2}$ in (2) and (3), we have

$X^2 = P^2 + R^2 + 2PR \ldots \ldots \ldots (5)$

$Y^2 = Q^2 + R^2 + 2QR \ldots \ldots \ldots (6)$

(5) – (6) gives

$X^2 - Y^2 = P^2 - Q^2 + PR - QR$

$= (P - Q)(P + Q + R)$

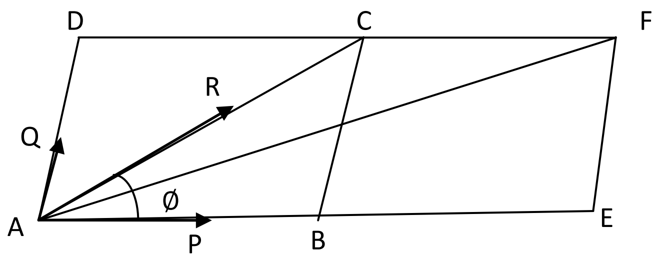
$= (P - Q).\ 0$

$= 0$

$\therefore X = Y.$

Example 1.2: If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle φ with the direction of P, show that the resultant of forces (P + R) and Q acting at the same angle will make an angle $\frac{\varphi}{2}$ with the direction of P + R.

First Method:



Let $\overline{AB}$ = P and $\overline{AD}$ = Q

From || gm ABCD,

$\overline{AB} + \overline{AD} = \overline{AC}$ = R.

To mark the force P + R,

produce AB to E so that BE = AC.

In the || gm DAEF,

$\overline{AF}$ gives the new resultant

In Δ CAF, CA = CF (each representing R in magnitude)

$\therefore \angle CAF = \angle CFA$

$= \angle FAE$ (alternate angles)

i.e. AF bisects $\angle CA$B

Second Method: The resultant of P + R and Q can be found in two stages. First, the resultant of P along AB and Q along AD is a force R along AC. Secondly, we have to find the resultant of the forces R along AC with an extra force R along AB. As these are equal, the final resultant bisects the angle BAC.

Check Your Progress

1. What is the resultant of forces?
2. What is the components of the forces?
3. What is the law of the parallelogram of forces?

1.7 Answers to Check Your Progress Questions

- If two or more forces F_1, F_2, F_3,....... etc. act on a rigid body and if a single force R can be found whose effect on the body is the same as that of all the forces F_1, F_2, F_3,.......... etc. put together, then the single force R is called the resultant of the forces F_1, F_2, F_3,....... etc
- The forces F_1, F_2, F_3,....... etc. are called the components of the force R.
- When two forces acting at a point are in different directions (i.e.) are inclined to each other, their resultant can be found with the help of a fundamental theorem in statics known as the law of the parallelogram of Forces.

1.8 SUMMARY

- When a number of forces act on a body and keep it at rest, the forces are said to be in equilibrium.
- If two forces be equal and opposite, i.e. if two forces acting on a body be such that they have (i) equal magnitude (ii) same line of action and (iii) opposite direction, then these two forces are in equilibrium.
- Conversely, if two forces acting on a body be in equilibrium, then they must be equal and opposite i.e. they must have (i) same magnitude (ii) same line of action and (iii) opposite directions.
- When two or more forces are acting together at a point it is always possible to find a single force which will have exactly the same effect as these forces. This single force is called the resultant force.

- Conversely, if two forces acting on a body be in equilibrium, then they must be equal and opposite i.e. they must have (i) same magnitude (ii) same line of action and (iii) opposite directions.
- When two or more forces are acting together at a point it is always possible to find a single force which will have exactly the same effect as these forces. This single force is called the resultant force.

1.9 KEY WORDS

- **Equilibrium:** When a number of forces act on a body and keep it at rest, the forces are said to be in equilibrium.
- **Equilibrium of two forces:** If two forces be equal and opposite, i.e. if two forces acting on a body be such that they have (i) equal magnitude (ii) same line of action and (iii) opposite direction, then these two forces are in equilibrium. Conversely if two forces acting on a body be in equilibrium, then they must be equal and opposite i.e. they must have (i) same magnitude (ii) same line of action and (iii) opposite directions.
- **Resultant force:** When two or more forces are acting together at a point it is always possible to find a single force which will have exactly the same effect as these forces. This single force is called the resultant force.

1.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Two forces of given magnitudes P and Q act at a point at an angle α. What will be (i) the maximum (ii) minimum value of resultant
2. The greatest and least magnitudes of the resultant of two forces of constant magnitudes are R and S respectively. Prove that, when the forces act at an angle 2φ, the resultant is of magnitude $\sqrt{R^2cos^2\varphi + S^2sin^2}\varphi$
3. The resultant of two forces P and Q is at right angles to P. Show that the angle between the forces is $\cos^{-1}(-\frac{P}{Q})$.
4. The resultant of two forces P and Q is of magnitude P. Show that, if P be doubled, the new resultant is at right angles to Q and its magnitude will be $\sqrt{4P^2 - Q^2}$.
5. Two equal forces act on a particle; find the angle between them when the square of their resultant is equal to three times their product
6. Two equal forces are inclined at an angle 2θ. Their resultant is their 3 times as great as when they are inclined at an angle 2φ. Show that $cos\theta = 3\ cos\varphi$.
7. The resultant of two forces P and Q is R. If Q be doubled, R is doubled. R is also doubled if Q is reversed.
   Show that $R' + R'^2 = 2(P^2+Q^2)$

1.11 FURTHER READINGS

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UNIT-II TRIANGLE OF FORCES

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2.0 INTRODUCTION

In this chapter we shall consider, the forces on a body with which we are chiefly concerned in statics can be classified as follows: (i) An attraction (ii) a tension and (iii) a reaction. We shall describe briefly these subdivisions of a force if three forces acting at a point are in equilibrium, then they can be represented in magnitude as well as direction by the three sides of a triangle taken in order such that its sides are parallel to the direction of the forces respectively. These are forces acting between two bodies which are not necessarily connected. When the bodies tent to approach each other, the force is called attraction and when they tent to separate out, the force is repulsion. Such forces are exerted without any visible means.

2.1 OBJECTIVES

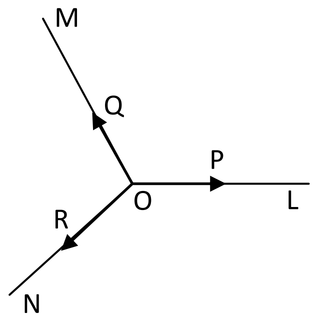
After going through this unit, you will be able to:

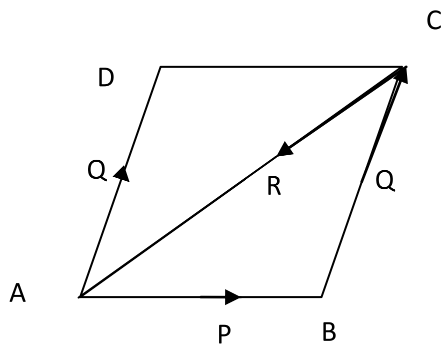
- Understand what triangle of forces is
- Explain about perpendicular triangle of forces
- Discuss triangle and resultant of force examples

2.2 TRIANGLE OF FORCES

The simple deduction from the parallelogram of forces is the following theorem, known as the Triangle of forces.

If the three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.





Let the forces, P, Q, R act at a point O and be represented in magnitude and direction by the sides AB, BC, CA of the triangle ABC. We have to prove that they will be in equilibrium.

Complete the parallelogram BADC. As AD is equal and parallel to BC, AD also represents Q in magnitude and direction.

P + Q $=\overline{AB} + \overline{AD}$
$= \overline{AC}$ (by || gm law.)

This shows that the resultant of the forces P and Q at O is represented in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

Hence P + Q + R = $\overline{AC}$ at O + $\overline{CA}$ at O
= $\overline{0}$ (as the two vectors at O are equal and opposite)

∴ The forces are in equilibrium.

Important note: In the above theorem, the forces P, Q, R are represented by the sides of the triangle ABC only in magnitude and direction but not in position. The forces act at a point and do not act along the sides of the triangle.

Corollary:

From the proof of the above theorem, it is clear that the resultant of the forces represented in magnitude and direction by the two sides AB and BC of the triangle ABC, is represented in magnitude and direction by AC.

This principle is stated as follows:

Notes

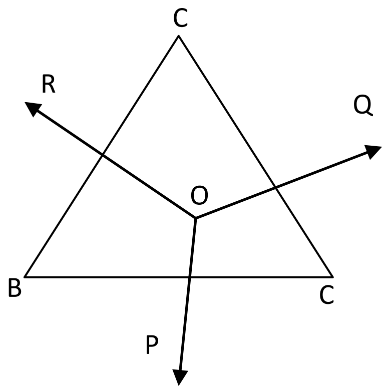
If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reserve order.

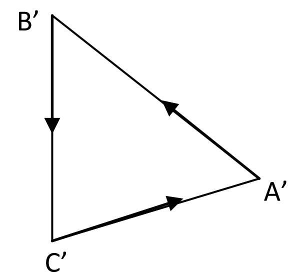
In the notation of vectors, the above means that

$$\overline{AB} + \overline{BC} = \overline{AC}$$

2.3 PERPENDICULAR TRIANGLE OF FORCES

If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.





Let the forces P, Q, R meet at O.
ABC is a triangle such that magnitudes of P, Q, R are proportional to the sides BC, CA and AB respectively of ΔABC and their directions are perpendicular to the corresponding sides all outwards.

We have to prove that they will be in equilibrium.

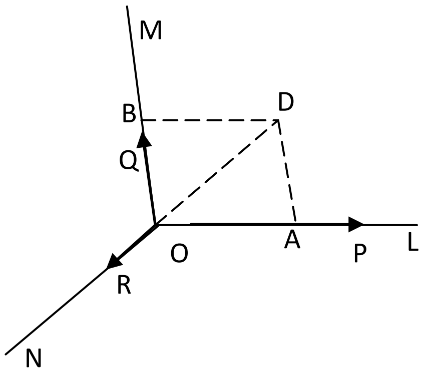
If we rotate the ΔABC through 90° in its own plane, we will get a new triangle A'B'C' whose sides are parallel to the given forces and represent the forces both in magnitude and direction.

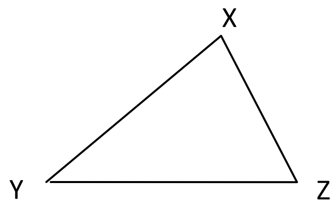
Hence by the triangle of forces, P, Q, R are in equilibrium.

Note: The above result will also be true, if the directions of the forces, instead of being perpendicular to the corresponding sides, make equal angles in the sense with them. The proof is exactly similar.

2.4 CONVERSE OF THE TRIANGLE OF FORCES

If three forces acting at a point are in equilibrium, then any triangle drawn so as to have its sides parallel to the directions of the forces shall represent them in magnitude also.





Let the three forces P, Q, R acting at O along the directions OL, OM and ON keep it in equilibrium. XYZ is triangle such that the sides YZ, ZX and XY are parallel to the directions of P, Q, R respectively. We have to prove that the sides of ΔXYZ are proportional to the magnitudes of P, Q and R given that P + Q + R = 0 (statically).

Along OL, cut off OA to represent the magnitude of P on some scale.

i.e. let $\overline{OA}$ = P

On the same scale, make $\overline{OB}$ = Q.

To get the resultant of P and Q, complete the || gm AOB.

The P + Q = $\overline{OA}$ + $\overline{OB}$ = $\overline{OD}$.

But P + Q + R = $\bar{0}$

i.e. $\overline{OD}$ + R = $\bar{0}$ or R = $\overline{DO}$

This shows that the third force R is represented in magnitude on the same scale by DO and that DON is a straight line.

Hence the three forces P, Q and R are parallel and proportional to the sides of the triangle OAD.

Now any triangle like XYZ whose sides are parallel to the directions of P, Q and R will be similar to ΔOAD and hence

$$\frac{YZ}{\mathrm{OA}} = \frac{ZX}{\mathrm{AD}} = \frac{XY}{\mathrm{DO}}$$

But $\frac{P}{\mathrm{OA}} = \frac{Q}{\mathrm{OB}} = \frac{R}{\mathrm{DO}}$;

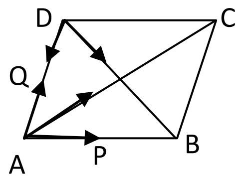
$$\therefore \frac{YZ}{P} = \frac{ZX}{\mathrm{Q}} = \frac{XY}{\mathrm{R}}$$

i.e. The sides ΔXYZ will be proportional to P, Q, R.

2.5 WORKED EXAMPLES

Example 2.1:

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.



Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD. Complete the parallelogram BAD.

Then P + Q = $\overline{AB} + \overline{AD} = \overline{AC}$ (|| gm law)

∴ $\overline{AC}$ is the sum of the two forces.

P - Q = $\overline{AB} - \overline{AD}$

= $\overline{AB} + \overline{DA}$

= $\overline{DA} + \overline{AB}$

= $\overline{DB}$ (by triangle law)

∴ $\overline{DB}$ is the difference of the two forces.

It is given that $\overline{AC}$ and $\overline{DB}$ are at right angles.

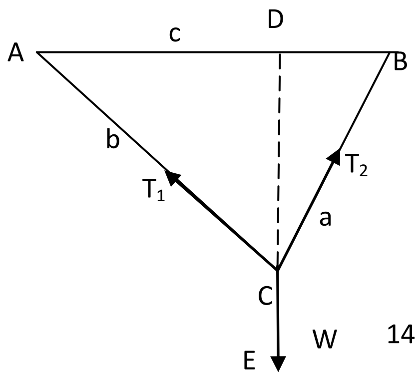
i.e. In parallelogram ABCD, the diagonals AC and BD cut at right angles.

∴ ABCD must be rhombus.

∴ AB = AD i.e. P = Q in magnitude.

Example 2.2:

A and B are two fixed points on a horizontal line at a distance c apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio $b(a^2 + c^2 - b^2) : a(b^2 + c^2 - a^2)$



Let T_1 and T_2 be the tensions along the strings CA and CB and W, the weight of the mass at C, acting vertically downwards along CE.

Produce EC to meet AB at D.

Since C is at rest under the action of the three forces, we have by Lemi's theorem,

$$\frac{T1}{\sin \angle ECB} = \frac{T2}{\sin \angle ECA} \;\ldots\ldots..(1)$$

Now $\sin \angle ECB = \sin(180° - \angle DCB)$

$= \sin \angle DCB$

$= \sin(90° - \angle ABC)$

$= \cos \angle ABC$

$\sin \angle ECA = \sin(180° - \angle ACD)$

$= \sin \angle ACD$

$= \sin(90° - \angle BAC)$

$= \cos \angle BAC$

Hence (1) becomes

$$\frac{T1}{\cos \angle ABC} = \frac{T2}{\cos \angle BAC}$$

$$\frac{T1}{T2} = \frac{\cos \angle ABC}{\cos \angle BAC} = \frac{\cos B}{\cos A} \qquad \ldots\ldots..(2)$$

In ΔABC, we know that

Cos B $=\frac{a^2 + c^2 - b^2}{2ca}$ and Cos A $=\frac{b^2 + c^2 - a^2}{2bc}$

Hence (2) becomes

$$\frac{T1}{T2} = \left(\frac{a^2 + c^2 - b^2}{2ca}\right) . \left(\frac{2bc}{b^2 + c^2 - a^2}\right)$$

$$= \frac{b(a^2 + c^2 - b^2)}{a(b^2 + c^2 - a^2)}$$

Check Your Progress

1. What is the triangle of forces?
2. What is the perpendicular of forces?
3. What is the converse of triangle of forces?
4. What is the perpendicular triangle?

2.6 ANSWER TO CHECK YOUR PROGRESS QUESTION

1. If the three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.
2. If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.
3. If three forces acting at a point are in equilibrium, then they can be represented in magnitude as well as direction by the three sides of a triangle taken in order such that its sides are parallel to the direction of the forces respectively.
4. The perpendicular from the vertex to the base line (the height) in an isosceles triangle divides the triangle into two equal right angled triangles. The sides of a right angled triangle ABC satisfy Pythagoras' rule, that is $a^2 + b^2 = c^2$. Also the converse is true.

2.7 SUMMARY

- The forces P, Q, R are represented by the sides of the triangle ABC only in magnitude and direction but not in position. The forces act at a point and do not act along the sides of the triangle.
- If the directions of the forces, instead of being perpendicular to the corresponding sides, make equal angles in the sense with them. The proof is exactly similar.
- If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.
- The perpendicular from the vertex to the base line (the height) in an isosceles triangle divides the triangle into two equal right angled triangles. The sides of a right angled triangle ABC satisfy Pythagoras' rule, that is $a^2 + b^2 = c^2$. Also the converse is true.
- If three forces acting at a point are in equilibrium, then they can be represented in magnitude as well as direction by the three sides of a triangle taken in order such that its sides are parallel to the direction of the forces respectively.

2.8 KEY WORDS

- **Converse of triangle:** If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.
- **Perpendicular triangle:** The perpendicular from the vertex to the base line (the height) in an isosceles triangle divides the triangle into two equal right angled triangles. The sides of a right angled triangle ABC satisfy Pythagoras' rule, that is $a^2 + b^2 = c^2$. Also the converse is true.

- **Triangle of forces:** If the three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium

2.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

1, Two forces act at a point and are such that if the direction of one is reversed, the direction of the resultant is turned through a right angle. Prove that the two forces must be equal in equilibrium.

2. Three forces X, Y, Z acting at a vertices A, B, C respectively of a triangle, each $\perp$ to the opposite side, keep it in equilibrium.

$$\text{Prove that } \frac{X}{a} = \frac{Y}{b} = \frac{Z}{c}$$

3. A weight is suspended by means of two equal strings attached to two points in a horizontal line. Show that if the lengths of the strings are increased, their tension is diminished.

4. A string ACB has its extremities tied to two fixed points A and B in the same horizontal line. To a given point C in the string, is knotted a given weight W. Find the tension of the string CA in the from $\frac{Wb}{4c\Delta}(c^2+a^2-b^2)$ where Δ is the area and a, b, c are the sides of the ΔABC.

5. If the tree forces represented in magnitude and direction by the bisectors of the angles of a triangle, all acting from the vertices be in equilibrium, show that the triangle must be equilateral.

2.10 FURTHER READINGS

Dr. M. K. Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.

Dr. M. K. Venkataraman, Dynamics, Agasthiar Publications, 13th Edition, 2009.

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UNIT III POLYGON OF FORCES

STRUCTURE

3.0 Introduction

3.1 Objectives

3.2 The Polygon of Forces

3.3 Lami's Theorem

3.4 An Extended form of the parallelogram law of forces

3.5 Worked examples

3.6 Answers to Check Your Progress Questions

3.7 Summary

3.8 Keywords

3.9 Self Assessment Questions and Exercises

3.10 Further Readings

3.0 INTRODUCTION

In this chapter we shall consider, Polygon law of vector addition states that if a number of vectors can be represented in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order. Lami's theorem states that if three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two forces. Consider three forces A, B, C acting on a particle or rigid body making angles α, β and γ with each other. If a body is subjected to many forces on its plane at a single point then they are called as Coplanar Concurrent Forces. The effect of the forces acting on the body is unknown. It is necessary to determine the resultant force of the coplanar forces to know this effect. Copy the link given below and paste it in new browser window to get more information on Law Of Polygon

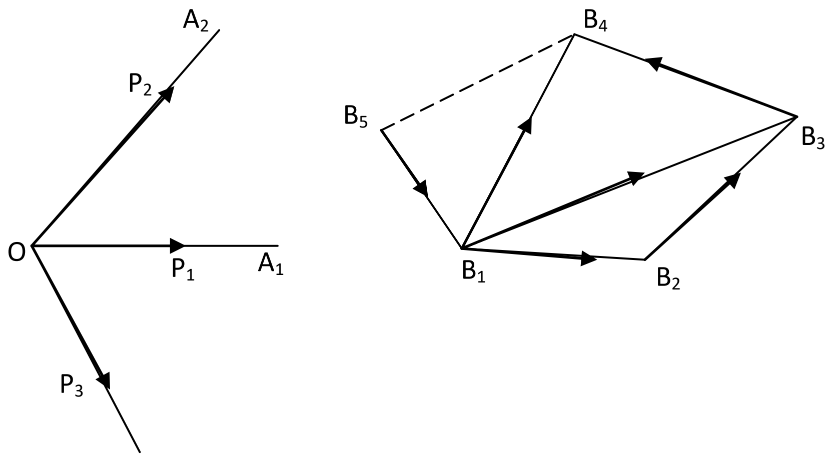
3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what polygon of forces is
- Explain about Lami's theorem
- Discuss polygon of forces and Lami's theorem examples

3.2 THE POLYGON OF FORCES

If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.



Let the forces P_1, P_2, ... P_n acting at O be represented in magnitude and direction by the sides B_1B_2, B_2B_3, ...B_nB_1 of the polygon $B_1B_2B_3$.........B_n.

We have to prove that the forces will be equilibrium.

Compounding the forces by vector law, step by step, we have

$P_1 + P_2 = \overline{B_1B_2} + \overline{B_2B_3} = \overline{B_1B_2}$

$P_1 + P_2 + P_3 = \overline{B_1B_3} + \overline{B_3B_4} = \overline{B_1B_4}$

And $P_1 + P_2 + P_3 + ...+ P_{n-1} = \overline{B_1B_{n-1}} + \overline{B_{n-1}B_n} = \overline{B_1B_n}$

It is to be noted that in each of the equilibrium above, the resultant on the right side, of the forces named on the left side, acts at the point O.

The last force P_n is represented by $\overline{B_nB_1}$

$\therefore P_1 + P_2 + P_3 +.....+ P_{n-1} + P_n = \overline{B_1B_n}$ at O + $\overline{B_nB_1}$ at O

$= \overline{0}$

$\therefore$ The forces are in equilibrium.

Note 1:

The above theorem is true even when the forces acting at O are not inb the same plane

Note 2:

The converse of the Polygon of Forces is not true. The converse of the triangle of forces is true because whenever the directions of three forces acting at a point and keeping it in equilibrium are known, all triangles drawn with their sides parallel to these directions, will be similar and hence represent the forces in magnitude also. But in the case of more than three forces acting at a point and keeping it in equilibrium, we cannot say that the sides of any polygon drawn with its sides parallel to the directions of the forces shall represent them in magnitude also. If we draw two such polygons, they will be merely equiangular and not necessary similar. All that we can say is that a polygon can be drawn with the sides parallel and proportional to the forces.

3.4 LAMI'S THEOREM

Father Lami gave the converse of the triangle of forces in the following trigonometrical form:

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.

We have proved that the sides of the triangle OAD represent the forces P, Q, R in magnitude and direction.

Applying the sine rule to ΔOAD, we have

$$\frac{OA}{\sin\angle ODA} = \frac{AD}{\sin\angle DOA} = \frac{DO}{\sin\angle OAD} \qquad \text{... ... .(1)}$$

But $\angle ODA$ = alt. $\angle BOD$ = 180° - $\angle MON$

$\therefore \sin\angle ODA = \sin(180° - \angle MON) = \sin\angle MON$ … …. ....(2)

Also $\angle DOA$ = 180° - $\angle NOL$

$\therefore \sin\angle DOA = \sin(180° - \angle NOL) = \sin\angle NOL$ ....(3)

and $\angle OAD$ = 180° - $\angle BOA$ = 180° - $\angle LOM$

$\therefore \sin\angle OAD = \sin(180° - \angle LOM) = \sin\angle LOM$ ....(4)

Substituting (2), (3), (4) in (1), we have

$$\frac{OA}{\sin\angle MON} = \frac{AD}{\sin\angle NOL} = \frac{DO}{\sin\angle LOM}$$

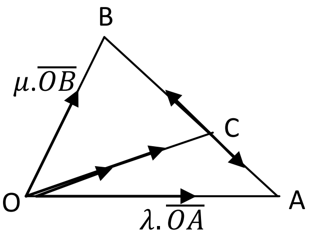
i.e. $\frac{P}{\sin\angle MON} = \frac{Q}{\sin\angle NOL} = \frac{R}{\sin\angle LOM}$

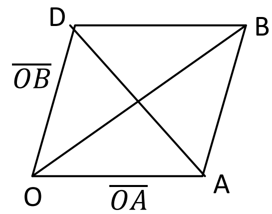
or $\frac{P}{\sin(Q,R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$

3.5 AN EXTENDED FORM OF THE PARALLELOGRAM LAW OF FORCES: THEOREM

If forces $\lambda\overline{OA}$ and $\mu\overline{OB}$ act at a point O along the lines OA and OB, their resultant will be the forces $(\lambda + \mu)\,\overline{OC}$ where C is the point on AB such that

$\lambda. AC = \mu. CB.$





Forces represented by $\lambda. OA$ and $\mu. OB$ act along the lines OA and OB. Take the point C on AB such that $\lambda. AC = \mu. CB$

From ΔOCA, $\overline{OA} = \overline{OC} + \overline{CA}$

$\therefore \lambda.\overline{OA} = \lambda.\overline{OC} + \lambda.\overline{CA}$ ... ...(1)

From ΔOCB, $\overline{OB} = \overline{OC} + \overline{CB}$

$\therefore \mu.\overline{OB} = \mu.\overline{OC} + \mu.\overline{CB}$ ... ...(2)

Adding (1) and (2),

$\lambda.\overline{OA} + \mu.\overline{OB} = (\lambda + \mu).\overline{OC} + \lambda.\overline{CA} + \mu.\overline{CB}$ ... ...(3)

By construction of the point C, we have $\lambda. AC = \mu. CB$.

$\therefore$ The forces $\lambda.\overline{CA}$ and $\mu.\overline{CB}$ are equal and opposite forces acting at C.

$\therefore \lambda.\overline{CA} + \mu.\overline{CB} = \overline{0}$

Hence (3) gives

$\lambda.\overline{OA} + \mu.\overline{OB} = (\lambda + \mu).\overline{OC}$ ... ...(4)

Important Corollary:

If we put $\lambda = 1 = \mu$, C becomes the midpoint of AB.

Then (4) gives,

$\overline{OA} + \overline{OB} = 2\,\overline{OC}$ ... ...(5)

i.e. The resultant of two forces represented completely by $\overline{OA}$ and $\overline{OB}$ is represented by $2\overline{OC}$, where C is the middle point of AB.

This result which will be greatly useful in the solution of a number of problems is also obvious from the parallelogram law since $\overline{OA} + \overline{OB} = \overline{OD}$

(Refer to fig above in which AOBD is a || gm)

C is the midpoint of the diagonal OD and so $\overline{OD} = 2\,\overline{OC}$.

3.6 WORKED EXAMPLES

Example 3.1:

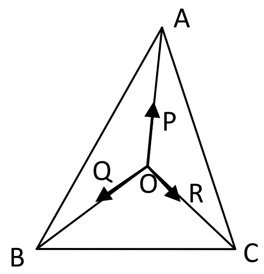
ABC is a given triangle. Forces P, Q, R acting along the lines OA, OB, OC are in equilibrium. Prove that

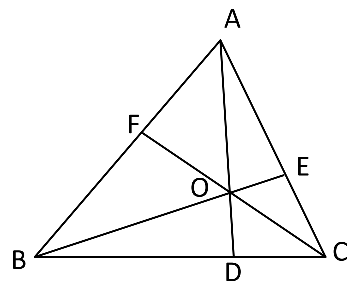
(i) P:Q:R = $a^2(b^2 + c^2 - a^2)$: $b^2(a^2 + c^2 - b^2)$: $c^2(a^2 + b^2 - c^2)$ if O is the circumcentre of the triangle.

(ii) P:Q:R = $\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$ if O is the in - centre of the triangle.

(iii) P:Q:R = a : b : c if O is the ortho centre of the triangle.

(iv) P:Q:R = OA : OB : OC if O is the centriod of the triangle,





By Lemi's theorem, we have

$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle COA} = \frac{R}{\sin\angle AOB} \qquad \ldots\ldots\ldots(1)$$

(i) When O is the circumcentre of the ΔABC,

$\angle BOC$ = 2 $\angle BAC$ = 2A; $\angle COA$ = 2 and $\angle AOB$ = 2C

∴ (1) gives $\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$

$$\therefore \frac{P}{2\sin A\cos A} = \frac{Q}{2\sin B\cos B} = \frac{R}{2\sin C\cos C} \qquad \ldots\ldots\ldots(2)$$

But Cos A =$\frac{b^2 + c^2 - a^2}{2bc}$ and sin A =$\frac{2\Delta}{bc}$, where Δ is the area.

$$\therefore\ 2\sin A\cos A = 2\ \frac{2\Delta}{bc}\ \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{2\Delta(b^2 + c^2 - a^2)}{b2c2}$$

Similarly 2 sin B cos B = $\frac{2\Delta(a^2 + c^2 - b^2)}{c2a2}$

and 2 sin C cos C = $\frac{2\Delta(a^2 + b^2 - c^2)}{a2b2}$

So (2) becomes

$$\frac{P.b^2c^2}{2\Delta(b^2 + c^2 - a^2)} = \frac{Q.c^2a^2}{2\Delta(a^2 + c^2 - b^2)} = \frac{R.a^2b^2}{2\Delta(a^2 + b^2 - c^2)}$$

Multiplying throughout by $\frac{2\Delta}{a2b2c2}$, we get

$$\frac{P}{a2(b^2+c^2-a^2)} = \frac{Q}{b2(a^2+c^2-b^2)} = \frac{R}{c2(a^2+b^2-c^2)}$$

(ii) When O is the in- centre of the triangle,

OB and OC are the bisectors of $\angle B$ and $\angle C$.

$\angle BOC = 180° - \frac{B}{2} - \frac{C}{2} = 180° - (\frac{B}{2} + \frac{C}{2})$

$= 180° - (90° - \frac{A}{2})$

$= 90° + \frac{A}{2}$

Similarly $\angle COA = 90° + \frac{B}{2}$ and $\angle AOB = 90° + \frac{C}{2}$

So (i) becomes

$$\frac{P}{\sin(90° + \frac{A}{2})} = \frac{Q}{\sin(90° + \frac{B}{2})} = \frac{R}{\sin(90° + \frac{C}{2})}$$

i.e. $\frac{P}{\cos\frac{A}{2}} = \frac{Q}{\cos\frac{B}{2}} = \frac{R}{\cos\frac{C}{2}}$

(iii) Let O be the ortho-centre of the triangle.

AD, BE, CF are the altitudes.
Quadrilateral AFOE is cyclic.

($\because$ $\angle AFO + \angle AEO = 90° + 90° = 180°$)

$\angle FOE$+ A = 180° or $\angle FOE = 180° - A$

$\angle BOC$ = vertically opposite $\angle FOE = 180° - A$

Similarly $\angle COA = 180° - B$ and $\angle AOB = 180° - C$.

Hence (1) becomes

$$\frac{P}{\sin(180° - A)} = \frac{Q}{\sin(180° - B)} = \frac{R}{\sin(180° - C)}$$

i.e. $\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$

i.e. . $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$ (since . $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$)

When O is the centriod of the triangle,

we know that $\Delta BOC = \Delta COA = \Delta AOB$

Notes

and each = $\frac{1}{3}\ \Delta ABC$

$\Delta ABC = \frac{1}{2}$ OB. OC $\sin\angle BOC = \frac{1}{3}\ \Delta ABC$

$\therefore \sin\angle BOC = \frac{2\Delta ABC}{3\ \mathrm{OB.OC}}$

Similarly $\sin\angle COA = \frac{2\Delta ABC}{3\ \mathrm{OC.OA}}$ and $\sin\angle AOB = \frac{2\Delta ABC}{3\ \mathrm{OA.OB}}$

Hence (1) becomes

$$\frac{\mathrm{P.3\ OB.OC}}{2\Delta ABC} = \frac{Q.3\ \mathrm{OC.OA}}{2\Delta ABC} = \frac{\mathrm{R.3\ OA.OB}}{2\Delta ABC}$$

i.e. P. OB.OC = Q.OC.OA = R.OA.OB

Dividing by OA.OB.OC throughout,

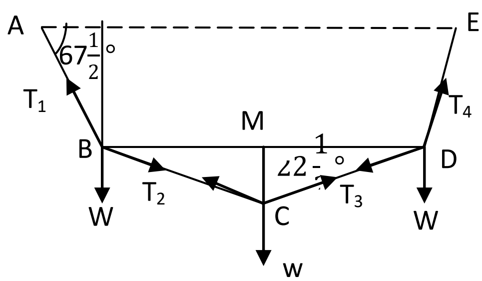
$\frac{\mathrm{P}}{\mathrm{OA}} = \frac{Q}{\mathrm{OB}} = \frac{R}{\mathrm{OC}}$

Example 3.2:

Weights W, w, W are the attached to points B, C, D respectively of a light string AE where B, C, D divide the string into 4 equal lengths. If the string hangs in the form of 4 consecutive sides of a regular octagon with the ends A and E attached to points on the same level, show that

$$W = (\sqrt{2} + 1)\ \mathrm{w}$$

ABCDE is a part of a regular octagon



We know that each interior angle of a regular polygon of n sides $= (\frac{2n-4}{n}) \times 90°$

Putting n = 8, each interior angle of ABCDE $= (\frac{2 \times 8 - 4}{8}) \times 90°$

$= \frac{12}{8} \times 90° = 135°$

Let the tensions in the portions AB, BC, CD, DE be T_1, T_2, T_3, T_4 respectively. The string BC pulls B towards C and pulls C towards B, the tensions being the same throughout its length. This fact is used to denote the forces acting at B, C and D.

In $\Delta BCD, \angle BCD = 135°$

$\therefore \angle CBD = \angle CDB = \frac{45°}{2} = 22\frac{1°}{2}$

$\therefore \angle ABD = \angle ABC - \angle CBD = 135° - 22\frac{1°}{2} = 112\frac{1°}{2}$

We know that every regular polygon is cyclic.

$\therefore$ A, B, C, D, E lie on the same circle.

$\therefore \angle EAB = 180° - \angle BDE$

$= 180° - \{\angle BDE - \angle BDC\}$

$= 180° - \{135° - 22\frac{1°}{2}\}$

$= 45° + 22\frac{1°}{2} = 67\frac{1°}{2}$

$\therefore \angle EAB + \angle ABD = 67\frac{1°}{2} + 112\frac{1°}{2} = 180°$.

$\therefore$ AE || BD.

$\therefore$ BD also is horizontal

Let the vertical line through B meet AE at L and the vertical line through C meet BD at M.

Applying Lami's theorem for the three forces at B, we get

$$\frac{W}{\sin \angle ABC} = \frac{T_2}{\sin(180° - \angle ABL)}$$

i.e. $$\frac{W}{\sin 135°} = \frac{T_2}{\sin \angle ABL} = \frac{T_2}{\sin 22\frac{1°}{2}}$$

($\because$ in rt. $\angle d \Delta ABL, \angle ABL = 180° - 67\frac{1°}{2}$)

$\therefore T_2 = \frac{W}{\sin 135°} \sin 22\frac{1°}{2}$ ....(1)

Similarly applying Lami's theorem for the three forces at C,

We have $$\frac{w}{\sin \angle BCD} = \frac{T_2}{\sin(180° - \angle MCD)}$$

i.e. $\frac{w}{\sin\ 135°} = \frac{T_2}{\sin\ \angle MCD} = \frac{T_2}{\sin\,(90°-22\,\frac{1°}{2}\,)} = \frac{T_2}{\cos 22\,\frac{1°}{2}}$

($\because$ in rt. $\angle d\Delta ABL, \angle ABL = 180° - 67\frac{1°}{2}$)

$\therefore T_2 = \frac{w}{\sin\ 135°}\cos 22\,\frac{1°}{2}$ ....(2)

Equating the two values of T_2 from (1) and (2), we have

$\frac{W}{\sin\ 135°}\sin 22\,\frac{1°}{2} = \frac{w}{\sin\ 135°}\cos 22\,\frac{1°}{2}$

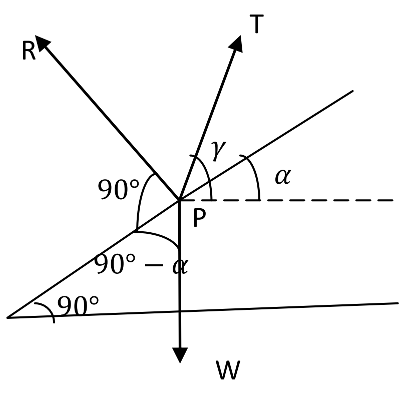
i.e. . $\frac{w}{W} = \tan 22\,\frac{1°}{2} = \sqrt{2} - 1$

$\therefore W = \frac{w}{\sqrt{2}-1} = \frac{w\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{w\sqrt{2}+1}{1}$

$= w(\sqrt{2}+1)$

Example 3.3:

A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ. If the slope of the plane be increased to β and the slope of the string unaltered, the tension of the string is doubled. Prove that $\cos\alpha - 2\cos\beta = \tan\lambda$



P is the position of the weight. The forces acting at P are (i) its weight W downwards (ii) the normal reaction R perpendicular to the inclined plane and (iii) the tension T along the string at an angle γ to the horizontal. By Lami's theorem for the three forces at P,

$\frac{T}{\sin\,(180° - \alpha\,)} = \frac{W}{\sin\,[90° - (\,\gamma - \alpha\,)]}$

i.e. $\frac{T}{\sin\ \alpha} = \frac{W}{\cos\,(\,\gamma - \alpha\,)}$

$\therefore \quad T \quad = \quad \frac{W\sin\ \alpha}{\cos\,(\,\gamma-\alpha\,)}$ ....(1)

In the second case, the inclination of the plane is β.

There is no change in γ.

If T_1 is the tension in the string, we will have

$T_1 = \frac{W\sin\ \beta}{\cos\,(\,\gamma-\beta\,)}$ ... ...(2)

But $T_2 = 2T$ (given)

$$\therefore \frac{W \sin\beta}{\cos(\gamma-\beta)} = \frac{2W \sin\alpha}{\cos(\gamma-\alpha)}$$

$\therefore$ $\sin\beta \cos(\gamma-\alpha) = 2\sin\alpha \cos(\gamma-\beta)$.

i.e. $\sin\beta\,(\cos\gamma\cos\alpha + \sin\gamma\sin\alpha) = 2\sin\alpha\,(\cos\gamma\cos\beta + \sin\gamma\sin\beta)$.

$$\sin\beta\cos\gamma\cos\alpha = 2\sin\alpha\cos\gamma\cos\beta + \sin\alpha\sin\gamma\sin\beta$$

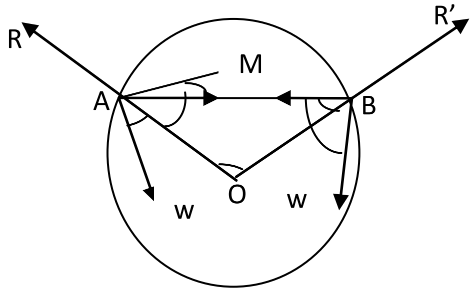
$$= \sin\alpha\,(2\cos\gamma\cos\beta + \sin\gamma\sin\beta)$$

$$\therefore \frac{\cos\alpha}{\sin\alpha} = \frac{2\cos\gamma\cos\beta + \sin\gamma\sin\beta}{\sin\beta\cos\gamma}$$

i.e. $\cot\alpha = 2\cot\beta + \tan\gamma$ or $\cot\alpha - 2\cot\beta = \tan\gamma$.

Example 3.4: Two beads of weights w and w' can slide on a smooth circular wire in a vertical plane. They are connected by a light string which subtends an angle 2β at the centre of the circle when the beads are in equilibrium on the upper half of the wire. Prove that the inclination of the string to the horizontal is given by

$$\tan\alpha = \frac{w \sim w'}{w + w'}\tan\beta$$



Let A and B be the beads of weights w and w' connected by a light string and sliding on a circular wire.

In equilibrium position, $\angle AOB = 2\beta$. O being the centre of the circle.

$$\therefore \angle OAB = \angle OBA = 90° - \beta.$$

Let AB make an angle α with the horizontal AN.

AL and BM are the vertical lines through A and B.

$\angle OAL = 90° - \angle OAN = 90° - (\angle OAB = \angle NAB)$

$$= 90° - (90° - \beta + \alpha)$$

$$= \beta - \alpha$$

Since AL || BM, $\angle ABM + \angle BAL = 180°$

$\therefore \angle ABM = 180° - \angle BAL = 180° - (90° - \alpha) = 90° + \alpha$

$\therefore \angle OBM = \angle ABM - \angle ABO$

$$= 90° + \alpha - (90° - \beta) = \alpha + \beta$$

The forces acting on the bead w at A are

[i] weight w acting vertically downwards along AL

[ii] normal reaction R due to contact with the wire along the radius OA outwards.

and [iii] tension T in the string along AB.

Similarly the forces acting on the bead w' at B are

[i] weight w' acting vertically downwards along BM

[ii] normal reaction R' due to contact with the wire along the radius OB outwards.

and [iii] tension T in the string along BA.

Applying Lami's theorem for the three forces at A,

$$\frac{w}{\sin[180° - (90° - \beta)]} = \frac{T}{\sin(180° - \beta - \alpha)}$$

i.e. $$\frac{w}{\cos\beta} = \frac{T}{\sin(\beta - \alpha)} \qquad \dots \dots \dots (1)$$

Similarly applying Lami's theorem for the three forces at B,

$$\frac{w'}{\sin[180° - (90° - \beta)]} = \frac{T}{\sin(180° - \beta + \alpha)}$$

i.e. $$\frac{w'}{\cos\beta} = \frac{T}{\sin(\beta + \alpha)} \qquad \dots \dots \dots (2)$$

Dividing (1) by (2) we have

$$\frac{w}{w'} = \frac{\sin(\beta + \alpha)}{\sin(\beta - \alpha)}$$

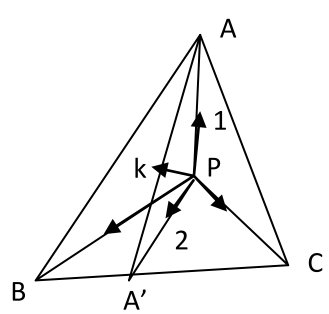
$$\frac{w - w'}{w + w'} = \frac{\sin(\beta + \alpha) - \sin(\beta - \alpha)}{\sin(\beta + \alpha) + \sin(\beta - \alpha)}$$

$$= \frac{2\cos\beta \sin\alpha}{2\sin\beta\cos\alpha} = \frac{\tan\alpha}{\tan\beta}$$

$$\tan\alpha = \frac{w - w'}{w + w'}\tan\beta.$$

Example 3.5:

ABC is a triangle. G is its centriod and P is any point in the plane of the triangle. Show that the resultant of forces represented by $\overline{PA}, \overline{PB}, \overline{PC}$ is 3 $\overline{PG}$ and find the position of P, if the three forces are in equilibrium.



Let A be the midpoint of BC.

Then $\overline{PB} + \overline{PC} = 2\,\overline{PA'}$

$\overline{PA} + \overline{PB} + \overline{PC} = \overline{PA} + 2\,\overline{PA'}$

$= 1.\,\overline{PA} + 2.\,\overline{PA'}$

$= (1+2)\,\overline{PK}$ ....(1)

where K is the point on AA' such that 1.AK=2.KA'

i.e. $\frac{AK}{KA'} = \frac{2}{1}$

i.e. K divides the median AA' in the ratio 2:1.

∴ K is the same as G, the centroid of the Δ.

∴ (1) becomes $\overline{PA} + \overline{PB} + \overline{PC} = 3\,\overline{PG}$.

If the three forces $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ are inequilibrium, then their resultant should be zero.

i.e. $\overline{PG} = \bar{0}$

∴ PG = 0

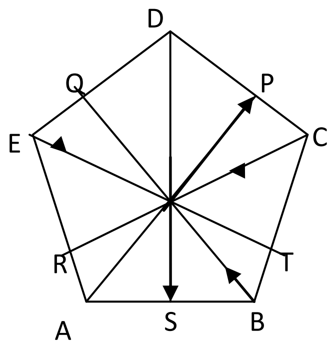
i.e. P must be taken at the centroid G of the triangle.

Note 3:

From the above, we find that if G is the centroid on a ΔABC, forces represented by GA, GB, GC will be in equilibrium.

i.e. $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$

This result is worth remembering.



Example 3.6:

Five forces acting at a point are represented in magnitude and direction by the lines joining the vertices of any pentagon to the midpoints of their opposite sides. Show that they are in equilibrium.

ABCDE is a pentagon and P, Q, R, S, T are the midpoints of the sides CD, DE, EA, AB and BC respectively.

$\overline{AP} + \overline{BQ} + \overline{CR} + \overline{DS} + \overline{ET} = \bar{0}$

Using the corollary of 10 we have,

$2\overline{AP} = (\overline{AC} + \overline{AD})$

i.e $\overline{AP} = \frac{1}{2}(\overline{AC} + \overline{AD})$

Similarly $\overline{BQ} = \frac{1}{2}(\overline{BD} + \overline{BE})$

$\overline{CR} = \frac{1}{2}(\overline{CE} + \overline{CA})$

$\overline{DS} = \frac{1}{2}(\overline{DA} + \overline{DB})$

and $\overline{ET} = \frac{1}{2}(\overline{EB} + \overline{EC})$

Adding up, we have

$\overline{AP} + \overline{BQ} + \overline{CR} + \overline{DS} + \overline{ET}$

$= \frac{1}{2}[\overline{AC} + \overline{AD} + \overline{BD} + \overline{BE} + \overline{CE} + \overline{CA}$

$+ \overline{DA} + \overline{DB} + \overline{EB} + \overline{EC}]$

$= \frac{1}{2} \times \overline{0}$

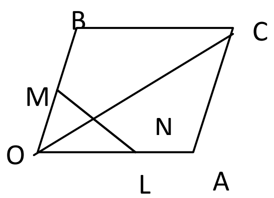
∵(the concerned vectors are equal and opposite)

$= \overline{0}$

Example 3.7:

OA, OB, OC are the lines of action of two forces P and Q and their resultant R respectively. Any transversal meets the lines in L, M and N respectively,

Prove that $\frac{P}{OL} + \frac{O}{OM} = \frac{R}{ON}$



Let $\overline{OA} =$ P and $\overline{OB}$ = Q. Complete the ∥ gm AOB.

$\overline{OC}$ = R.

Let $\frac{OA}{OL} = \lambda$ and $\frac{OB}{OM} = \mu$.

∴OA = λ.OL and OB = μ OM

∴ $\overline{OA} + \overline{OB} = \lambda\,\overline{OL} + \mu\,\overline{OM}$

$= (\lambda + \mu)\,\overline{OK}$ ....(1) where K is a point on LM.

But $\overline{OA} + \overline{OB} = \overline{OC}$ …....(2)

Hence the forces $(\lambda + \mu)\,\overline{OK}$ and $\overline{OC}$ must be the same.

i.e. K must be a point on OC also.

∴ K is the point of intersection of OC and LM.

i.e. K is clearly the same as N.

Equating the magnitudes of the two equal forces on the right hand sides of (1) and (2), we get

$(\lambda + \mu)$. OK = OC

i.e.$(\lambda + \mu)$. ON = OC

or $\lambda + \mu = \frac{OC}{ON}$

i.e. $\frac{OA}{OL} + \frac{OB}{OM} = \frac{OC}{ON}$

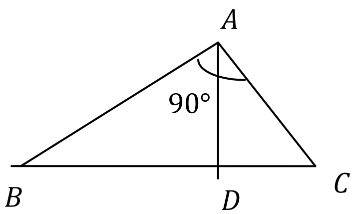
or $\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$

Example 3.8:

ABC is a triangle, with a right angle at A. AD is the perpendicular on BC. Prove that the resultant of the forces $\frac{1}{AB}$ acting along AB and $\frac{1}{Ac}$ acting along AC is $\frac{1}{AD}$ acting along AD.

From Geometry, we have the following well known results

AB^2 = BC. BD; AC^2 = BC. CD; AD^2 = BD. DC



The forces $\frac{1}{AB}$ acting along AB and $\frac{1}{AC}$ acting along AC can be considered respectively as the forces $\frac{1}{AB^2}$. AB acting along AB and $\frac{1}{AC^2}$. AC acting along AC.

If we take $\lambda = \frac{1}{AB^2}$ and $\mu = \frac{1}{AC^2}$

then $\lambda . AB^2 = \mu . AC^2$ (each being = 1)

i.e. $\lambda . BC . BD = \mu$ BC. CD

i.e. λ. BD = μ. CD

Hence the resultant of forces $\lambda . \overline{AB}$ and $\mu . \overline{AC}$ is the force $(\lambda + \mu). \overline{AD}$

i.e. The resultant of forces $\frac{1}{AB}$ along AB and $\frac{1}{AC}$ along AC is the force $(\lambda + \mu)$.AD acting along AD.

Magnitude of the resultant

$= (\lambda + \mu)$.AD

$= \left(\frac{1}{AB^2} + \frac{1}{AC^2}\right)$ AD

$= \left(\frac{AC^2 + AB^2}{AB^2 AC^2}\right)$ AD

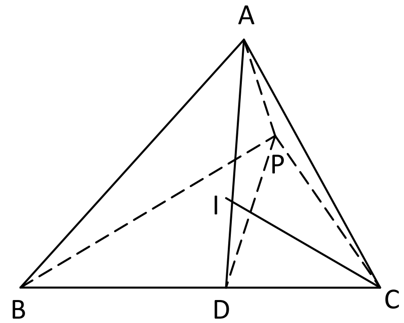
$= \frac{BC^2}{AB^2AC^2}$. AD = $\frac{BC^2}{BC.BD.BC.CD}$. AD

= $\frac{1}{BD.CD}$. AD = $\frac{1}{AD^2}.AD$ = $\frac{1}{AD}$

and this acts along AD.

Example 3.9:

P is a point in the plane of the triangle ABC and I is the incentre. Shoe that the resultant of forces represented by PA.sin A, PB. .sin B and PC.sin C is 4PI. .cos $\frac{A}{2}$. cos $\frac{B}{2}$ cos $\frac{C}{2}$



Let AD be the internal bisector of $\angle A$ and I the incentre.

Then we know that

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} = \frac{\sin C}{\sin B}$$

$\therefore$ sin B.BD = sin C.DC ... .(1)

Also, as BI bisects $\angle B$, $\frac{AI}{ID} = \frac{AB}{BD}$

As BI bisects $\angle C$, $\frac{AI}{ID} = \frac{AC}{CD}$

$\therefore$ $\frac{AI}{ID} = \frac{AB}{BD} = \frac{AC}{CD}$

$\therefore$ $\frac{AI}{ID}$ is also $\frac{AB+AC}{BD+CD} = \frac{AB+AC}{BC} = \frac{c+b}{a} = \frac{\sin B+\sin C}{\sin A}$

i.e. AI sin A = (sin B + sin C).ID ... ...(2)

Now,

$\overline{PB}$. sin B+ $\overline{PC}$ sin C = (sin B + sin C).$\overline{PD}$ ... ...(3)

[since sin $B.BD$=sin C. DC from (1)]

$\therefore$ $\overline{PA}$. sin A+ $\overline{PB}$. sin B+ $\overline{PC}$ sin C

= $\overline{PA}$. sin A (sin B + sin C).$\overline{PD}$

= (sin A + sin B + sin C).$\overline{PI}$ ... ...(4)

$$= (\sin A + \sin B + \sin C).\overline{PI} \ldots\ldots \ldots \ldots(4)$$

since AI.$\sin A = (\sin B + \sin C)$.ID from (2)

But we know that in a Δ,

$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ ....(5)

Using (5) in (4), we have

the required resultant = $4\cos\frac{A}{2}.\cos\frac{B}{2}\cos\frac{C}{2}.\overline{PI}$

Check Your Progress

1. What is the statement of Polygon of forces?
2. What is the statement of Lami's theorem?
3. What is the statement of Parallelogram of forces?

3.7 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.
2. If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.
3. If forces $\lambda\overline{OA}$ and $\mu\overline{OB}$ act at a point O along the lines OA and OB, their resultant will be the forces $(\lambda + \mu)\overline{OC}$ where C is the point on AB such that $\lambda.AC = \mu.CB$.

3.8 SUMMARY

- The resultant of two forces represented completely by $\overline{OA}$ and $\overline{OB}$ is represented by $2\overline{OC}$, where C is the middle point of AB.
- If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.
- If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.
- If forces $\lambda\overline{OA}$ and $\mu\overline{OB}$ act at a point O along the lines OA and OB, their resultant will be the forces $(\lambda + \mu)\overline{OC}$ where C is the point on AB such that $\lambda.AC = \mu.CB$.

3.9 KEYWORDS

- **Polygon of forces:** If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.

- **Lami's theorem:** If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.
- **Parallelogram of forces:** If forces $\lambda\overline{OA}$ and $\mu\overline{OB}$ act at a point O along the lines OA and OB, their resultant will be the forces $(\lambda + \mu)\,\overline{OC}$ where C is the point on AB such that $\lambda.AC = \mu.CB.$

3.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. A, B, C are three points on the circumference of a circle and forces P and Q act along AB and BC respectively. If their resultant is a force R along the tangent at B, Show that $\frac{P}{BC} = \frac{Q}{AB} = \frac{R}{AC}$.

2. A bead is free to slide on a smooth vertical circle and is connected by a string equal in length to the radius of the circle, to the highest point of the circle; find the tension of the string and the reaction of the circle.

3. The ends of a piece of string are attached to two heavy rings P and Q of weights W_1 and W_2 respectively which are free to slide on two smooth rods BA and BC respectively inclined at angle α and β to the horizon and lying in the same vertical plane. Show that in the position of equilibrium, the string is inclined to the horizon at an angle $tan^{-1}[\frac{W_1 \cot\beta \sim W_2 \cot\alpha}{W_1 + W_2}]$.

4. Show that the system of forces represented by the lines joining any point to the vertices of a triangle is equivalent to the system represented by the lines joining the same point to the middle points of the sides of the triangle.

5. The sides BC, CA, AB of a ΔABC are bisected in D, E, F respectively. Show that the forces represented by DA, EB and FC are in equilibrium.

6. Three forces PA, PB, PC diverge from the point P and three forces AQ, BQ, CQ converge to the point Q. Show that the resultant of the six forces is represented in magnitude and direction by 3PQ and that it passes through the centroid of the triangle ABC.

7. If ABCD is a parallelogram and P any point, show that the forces represented by PA and PC are equivalent to the forces represented by PB and PD.

8. If P is any point within the quadrilateral ABCD, find the resultant of the forces represented by PA, PB, PC, PD and hence find the position of the point P so that these forces may be in equilibrium.

9. Q is any point within the quadrilateral ABCD. P, Q, R, S are the midpoints of the sides. Show that the forces represented by OA, OB, OC, OD have the same resultant as the forces represented by OP, OQ, OR, OS.

10. PQRS is a quadrilateral. Prove that the resultant of the forces completely represented by the sides PQ, QR, PS, SR is represented in magnitude and direction by 2PR, and that its line of action bisects QS.

11. ABCD is quadrilateral and E is the point of intersection of the lines joining the middle points of the opposite sides. If O is any point in that plane, show that the resultant of the forces $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$ is equal to $4\overline{OE}$.

12. ABCDEF is a regular hexagon and O is any point. Prove that the resultant of forces represented by OA, OB, OC, OD, OE, OF is a force 6 OP, where P is the centre of the circumcircle of the hexagon.

13. ` ABC is a triangle. Find the resultant of the forces at A represented in magnitude and direction by
(i) 3AB and 2AC (ii) 3AB and 2CA

3.11 FURTHER READINGS

Dr. M. K. Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.

Dr. M. K. Venkataraman, Dynamics, Agasthiar Publications, 13th Edition, 2009.

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UNIT IV RESOLUTION OF FORCES

STRUCTURE

4.0 Introduction

4.1 Objectives

4.2 Resolution of a Forces

4.3Components of a Forces

4.4 Theorem on Resolved Parts

4.5 Resultant of any Number of Forces

4.6 Resultant of any Number of Coplanar Forces

4.7Condition of Equilibrium

4.8Worked examples

4.9 Answers to Check Your Progress Questions

4.10 Summary

4.11 Keywords

4.12 Self Assessment Questions and Exercises

4.13 Further Readings

4.0 INTRODUCTION

In this chapter we will consider, resolution of force F can be resolved into (or replaced by) two forces, which together produces the same effects that of force F. These forces are called the components of the force F. This process of replacing a force into its components is known as resolution of a force into components. A force can be resolved into two components, which are either perpendicular to each other or inclined to each other. If the two components are perpendicular to one another, then they are known as rectangular components and when the components are inclined to each other, they are called as inclined components. The force F can now be resolved into two components F_x and F_y along the x and y axes and hence, the components are called rectangular components. Further, the polygon constructed with these two components as adjacent sides will form a rectangle OABC and, therefore, the components are known as rectangular components.

4.1 OBJECTIVES

After going through this unit, you will be able to:

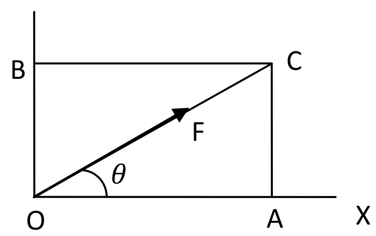
- Understand what resolution of forces is
- Explain about theorems on resolved parts
- Discuss resolution of forces and condition of equilibrium examples

4.2 RESOLUTION OF A FORCE

Two forces given in magnitude and direction have only one resultant, for we can construct only one parallelogram when two adjacent sides are given. Conversely, a single force can resolved into two components in an infinite number of ways, since any number of parallelograms can be constructed on a given line AC as diagonal. If BADC is any one of these, the force AC is equivalent to the two component forces AB and AD.

The most important case of resolution of a force occurs, when a given force is to be resolved in two directions at right angles, one of these directions being given. In this case, the magnitudes of the component forces are easily got as follows:

Let OC represent the given forces F and OX be a line inclined at an angle θ to OC. Let OY be perpendicular to OX. Draw CA $\perp$to OX and complete the parallelogram OACB. Then the force OC is equivalent to the two component forces OA and OB.

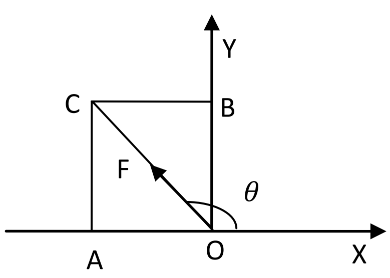


Also OA = OC. cosθ = F. cosθ

and OA = AC= OC. sinθ = F. sinθ

When a given force is resolved into two components in two mutually perpendicular directions, the components are referred to as the resolved parts in the corresponding directions.

OA is the resolved part of F along OX, while OB is the resolved part of F along OY.



θ is obtuse and OA is in a direction opposite to OX.
In this case, the resolved part of F along OX is negative. Its value as before is Fcosθ, which is negative, as θ is obtuse.

Hence we have the following important proposition:

A force F is equivalent to a force F $\cos\theta$ along a line making an angle θ with its own direction together with a force F $\sin\theta$ perpendicular to the direction of the first component.

Corollary 1.

When $\theta = 0$, $\cos 0 = 1$. The resolved part = F.

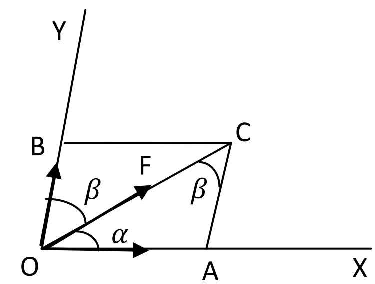
i.e. the resolved part of a force in its own direction is the force itself.

Corollary 2.

When $\theta = 90°$, $\cos 90° = 0$. The resolved part = 0.

i.e. A force has no resolved part in a direction perpendicular to itself.

4.3 COMPONENTS OF A FORCE ALONG TWO GIVEN DIRECTIONS:



Let OC represent a given force F and OX, OY be two lines making angle α and β with OC.

Draw CA parallel to OY and CB parallel to OX, making the parallelogram OACB as shown in the figure.

Then OA and OB are the components of the force OC along OX and OY respectively.

From $\Delta OAC,$

$$\frac{OA}{\sin \angle OCA} = \frac{AC}{\sin \angle AOc} = \frac{OC}{\sin \angle OAC}$$

i.e. $$\frac{\mathrm{OA}}{\sin \beta} = \frac{\mathrm{AC}}{\sin \alpha} = \frac{\mathrm{OC}}{\sin[180° - (\alpha + \beta)]}$$

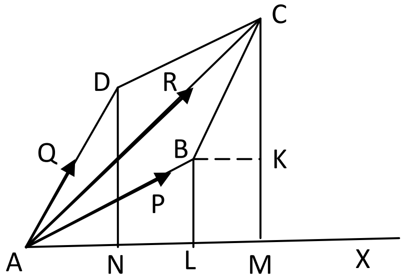
i.e. $$\frac{\mathrm{OA}}{\sin \beta} = \frac{\mathrm{AC}}{\sin \alpha} = \frac{F}{\sin(\alpha + \beta)}$$

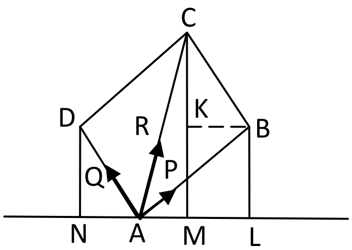
$\therefore$ OA = $\dfrac{F \sin \beta}{\sin(\alpha + \beta)}$ and OB = AC = $\dfrac{F \sin \alpha}{\sin(\alpha + \beta)}$

It should be noted that the component of a force in a given direction is different from the resolved part of the force in that direction. To find the component of a force in any direction, we must be given the direction of the other component also, On the other hand, to find the resolved part, we need only the given direction, since the other direction must be at right angles to it. In other hand, the component of a force in a direction is a variable quantity, while the resolved part of the force in a direction is a fixed quantity, its value being F cos θ as shown already.

4.4 THEOREM ON RESOLVED PARTS

The algebraic sum of the resolved parts of two forces in any direction is equal to the resolved part of the resultant in the same direction.





Let AB and AD represent completely the forces P and Q and AX be the direction in which the forces are to be resolved. Complete the parallelogram ABCD so that the resultant R is represented by AC.

Draw BL, DN and CM perpendiculars to OX and BK $\perp$to CM.

Then AL, AN and AM are the resolved parts of the forces P, Q, R along AX.

AD makes an obtuse angle with AX and so the resolved part of Q is -AN.

We have to show that AL $\pm$ AN = AM.

The triangles DAN and CBK are congruent and hence AN = BK.

AL $\pm$ AN = AL $\pm$ BK = AL $\pm$ LM = AM.

Obviously the above theorem can be extended to the resultant of any number of forces acting at a point.

Suppose P_1, P_2, P_3 are three forces acting at O.

Let R_1 be the resultant of P_1 and P_2 and R_2 be the resultant of R_1 and P_3.

Applying the theorem to the two sets of three forces P_1, P_2, R_1 and R_1, P_3, R_2, we have

resolved part of R_1 along OX = resolved part of P_1

+ resolved part of P_2 ....(1)

and resolved part of R_2 along OX = resolved part R_1

$_+$ resolved part of P_3 ... ...(2)

Combining (1) and (2), we have

resolved part of R_2 = resolved part of P_1 + resolved part of P_2

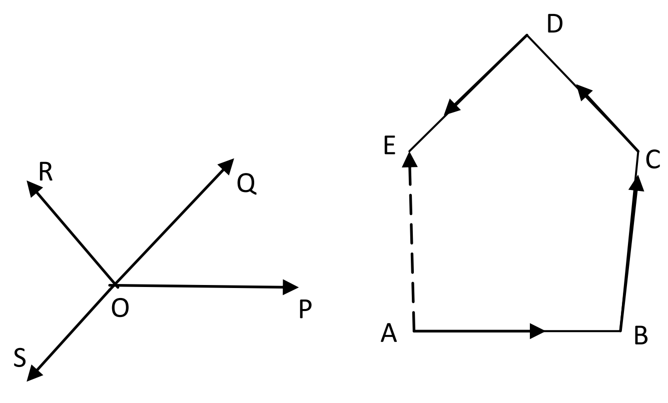
+ resolved part of P_3 and so on.

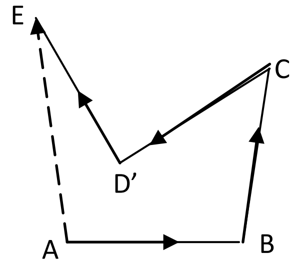
Hence in a generalized form, we have the theorem

The algebraic sum of the resolved parts of a number of forces in any direction is equal to the resolved part of the resultant in the same direction.

In the application of this theorem, it is to be noted that all the forces are resolved in the same direction and each resolved part has to be taken with its proper sign.

4.5 RESULTANT OF ANY NUMBER OF FORCES: GRAPHICAL METHOD





Let P, Q, R, S be the forces acting at O.

Take a point A and draw lines AB, BC, CD and DE to represent successively the forces P, Q, R and S in magnitude and direction.

Compounding the forces be vector law, step by step, we have

$$P + Q = \overline{AB} + \overline{AC} = \overline{AC}$$

$$P + Q + R = \overline{AC} + \overline{CD} = \overline{AD}$$

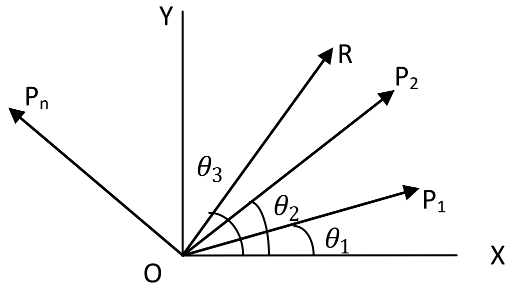
and $$P + Q + R + S = \overline{AD} + \overline{DE} = \overline{AE}$$

Hence the required resultant is represented is magnitude and direction by the line AE. The same construction will apply for any number of forces. The figure ABCDE is said to be the force-polygon.

The force-polygon can be constructed by drawing the vectors corresponding to the forces in any order. In fig 31, the order of the third and the fourth forces have been interchanged but AE is the same in each case.

4.6 RESULTANT OF ANY NUMBER OF COPLANAR FORCES

Let forces P_1, P_2, P_3, ... P_n act at O. Through O, draw two lines OX and OY at right angles to each other in the plane of the forces. Let the lines of action of P_1, P_2,... P_n make angles α_1, α_2, ... α_n with OX. Let R be the resultant inclined at angle θ to OX.



Then,

$R\cos\theta$ = resolved part of the resultant along OX

= algebraic sum of the resolved parts of

$P_1, P_2, \ldots P_n$ along OX.

$= P_1\cos\theta_1 + P_2\cos\theta_2 \ldots + P_n\cos\theta_n$

= X (say) ....(1)

$R\sin\theta$ = resolved part of the resultant along OY

= algebraic sum of the resolved parts of

$P_1, P_2, \ldots P_n$ along OY.

$= P_1\sin\theta_1 + P_2\sin\theta_2 \ldots + P_n\sin\theta_n$

= Y (say) ....(2)

Squaring (1) and (2) and adding, we have

$R^2 = X^2 + Y^2$

i.e. $R = \sqrt{X^2 + Y^2}$ ....(3)

Dividing (2) by (1), we get

$\tan\theta = \frac{Y}{X}$ i.e. $\theta = tan^{-1}(\frac{Y}{X})$ ....(4)

Equations (3) by (4) give respectively the magnitude and direction of the resultant.

4.7 CONDITION OF EQUILIBRIUM

Forces acting at a point are in equilibrium when their resultant is zero. We shall now give the conditions which must be satisfied by a number of forces acting at a point of a rigid body or on a particle, in order that the body, or the particle may be at rest.

Geometrical or graphical conditions:

We have already studied the Triangle of Forces and the Polygon of Forces.

If forces acting at a point are represented in magnitude and direction by lines forming the successive sides of a polygon, then for equilibrium, the polygon must be closed. When there are only three forces acting on a particle, the conditions of equilibrium are often most easily found by applying Lami's theorem.

Analytical Conditions:

If we resolve the forces in any two directions at right angles and the sums of the components in these directions be X and Y, the resultant R is given by $R^2 = X^2 + Y^2$.

If the forces are in equilibrium, R = 0.

Then $X^2 + Y^2 = 0$

Now, the sum of the squares of two real quantities cannot be zero unless each quantity is separately zero.

$\therefore$ X = 0 and Y = 0.

Hence, if any number of forces acting at a point are in equilibrium, the algebraic sums of the resolved parts of the forces in any two perpendicular directions must be zero separately.

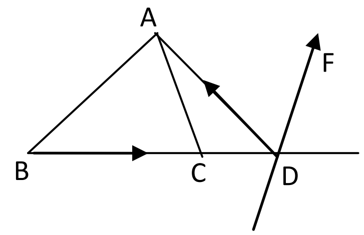
Conversely if the algebraic sum of the resolved parts of the forces acting at a point in any two perpendicular directions are zero separately, the forces will be in equilibrium.

This is because when X = 0 and Y = 0, we must have R = 0.

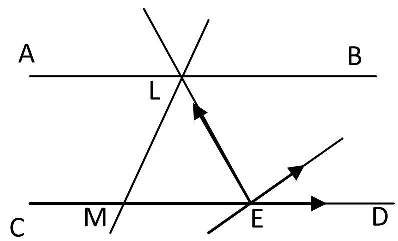
4.8 WORKED EXAMPLE

Example 4.1:

Show that a given force may be resolved into three components, acting in three given lines which are not all parallel or all concurrent.



Let the three lines form a ΔABC and let the given force F meet the side BC in D. Then F can be resolved into two components acting along BC and DA respectively. The component along DA can be resolved into two components along AB and AC respectively.



Suppose two of the lines AB and CD are parallel and LM is the third line. Let the given force F meet CD at E. F can be resolved into two components along CD and EL. The component along EL can be resolved into two components acting along BA and ML respectively.

Example 4.2:

ABCD is quadrilateral and forces acting at a point are represented in direction and magnitude by BA, BC, CD and DA. Find their resultant.

We have $\overline{BC} + \overline{CD} + \overline{DA} = \overline{BA}$

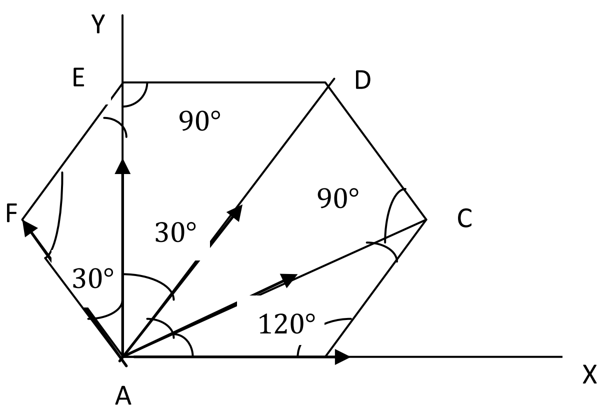
$\therefore\ \overline{BA} + (\overline{BC} + \overline{CD} + \overline{DA}) = \overline{BA} + \overline{BA} = 2\,\overline{BA}$

Hence the resultant is 2BA, both in magnitude and direction

Example 4.3:

ABCDEF is a regular hexagon and at A, act forces represented by $\overline{AB}$, $2\overline{AC}$, $3\overline{AD}$, $4\overline{AE}$ and $5\overline{AF}$. Shoe that the magnitude of the resultant is AB. $\sqrt{351}$ and that it makes an angle $tan^{-1}(\frac{7}{\sqrt{3}})$ with AB.

Let a be the side of the hexagon.



Each interior angle of a regular hexagon = 120°

$\therefore \angle CAB = \angle ACB = 30°$

$= \angle FAE = \angle FEA$

From the isosceles Δ ABC,

AC = 2AB.cos30°

$= 2a\frac{\sqrt{3}}{2} = a\sqrt{3}$ = AE

$\angle AED = 90°$ $\therefore AD^2 = AE^2 + ED^2 = 3a^2 + a^2 = 4a^2$

∴ AD = 2a

Since the vertices of a regular hexagon lie on a circle,

$\angle DAB = \angle DCB = 180°$

$\therefore \angle DAB = 180° - 120° = 60°$

$\therefore \angle DAB = 30°$ and $\angle EAD = 30°$.

The magnitudes of the forces acting at A are a, $2a\sqrt{3}$, 6a, $4a\sqrt{3}$ and 5a as shown in the figure.

Take AB and AE as axes of x and y and let R be the resultant inclined at an angle θ to AB.

Resolving the forces along AB and AE, we have

$R\cos\theta = a + 2a\sqrt{3}\cos 30° + 6a\cos 60° + 5a\cos 120°$

$= a + 2a\sqrt{3}\,\frac{\sqrt{3}}{2} + 6a.\frac{1}{2} - 5a.\frac{1}{2}$

$= \frac{9a}{2}$ .. ..(1)

and $R\sin\theta = 2a\sqrt{3}\cos 60° + 6a\cos 30° + 4a\sqrt{3} + 5a\cos 30°$

$= 5a\sqrt{3} + \frac{11a\sqrt{3}}{2}$

$= \frac{21a\sqrt{3}}{2}$ …………....(2)

Squaring (1) and (2), and adding,

$R^2 = (\frac{9a}{2})^2 + (\frac{21a\sqrt{3}}{2})^2 = \frac{81a^2}{4} + \frac{441}{4} \times 3a^2$

$= \frac{1404}{4} a^2 = 351a^2$

$\therefore$ R = a$\sqrt{351}$ = AB. $\sqrt{351}$

Dividing (2) by (1),

$\tan\theta = \frac{21a\sqrt{3}}{2} \times \frac{2}{9a} = \frac{21\sqrt{3}}{9} = \frac{7\sqrt{3}}{3} = \frac{7}{\sqrt{3}}$

Hence the resultant is a force of magnitude AB $\sqrt{351}$,

In a direction making an angle $tan^{-1}(\frac{7}{\sqrt{3}})$ with AB

Example 4.4:

Forces acting at a point are represented in magnitude and direction by $\overline{AB}$, $2\overline{BC}$, $2\overline{CD}$, $\overline{DA}$ and $\overline{DB}$ where ABCD is a square. Show that the forces are in equilibrium.

$\overline{AB} + 2\overline{BC} + 2\overline{CD} + \overline{DA} + \overline{DB}$

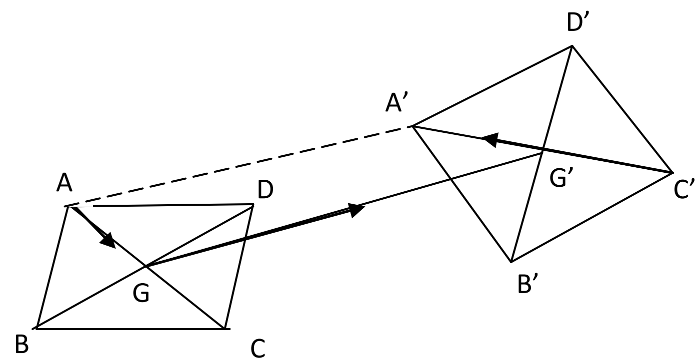
$= (\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}) + (\overline{BC} + \overline{CD} + \overline{DB})$

$= \overline{0}$ [since the forces in the first set of brackets are in equilibrium by the polygon of forces (square ABCD) and the forces in the second set are in equilibrium by the triangle of forces (triangle BCD)]

Hence the given set of forces are in equilibrium.

Example 4.5:

ABCD and $A'B'C'D'$ are parallelograms, prove that forces $\overline{AA'}$, $\overline{B'B}$, $\overline{CC'}$ and $\overline{D'D}$ acting at a point will keep it at rest.



Let G and G' be the points of intersection of the diagonals.

By the polygon of forces, from the quadrilateral AGG'A',

$\overline{AA'} = \overline{AG} + \overline{GG'} + \overline{G'A'}$

Similarly $\overline{B'B} = \overline{B'G'} + \overline{G'G} + \overline{G}$B

$\overline{CC'} = \overline{CG} + \overline{GG'} + \overline{G'C'}$

$\overline{D'D} = \overline{D'G'} + \overline{G'G} + \overline{G}$D

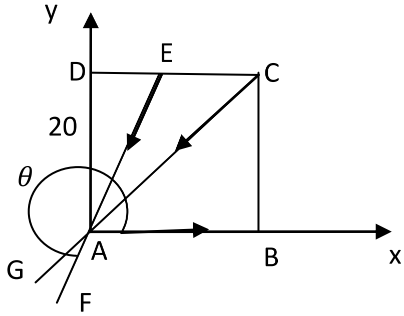
Adding up,

$\overline{AA'} + \overline{B'B} + \overline{CC'} + \overline{D'D} = \overline{0}$

($\therefore$ the concerned vectors in the right side are equal and opposite).

Example 4.6:

E is the middle point of the side CD of a square ABCD. Forces 16, 20, $4\sqrt{5}$, $12\sqrt{2}$ kg.wt. act along AB, AD, EA, CA in the directions indicated by the order of the letters. Show that they are in equilibrium.



Take AB and AD as axes of x and y.

Produce EA to F and let $\angle BAF = \theta$

Produce CA to G.

$\angle BAG = \angle BAC + \angle CAG$

$= 45° + 180° = 225°$

Let R be the resultant of the forces inclined at an angle θ to AB.

Resolving the forces along AB and AD, we have

$R\cos\theta = 16 + 12\sqrt{2}\cos 225° + 4\sqrt{5}\cos\theta$

$= 16 + 12\sqrt{2}\cos(180° + 45°) + 4\sqrt{5}\cos\theta$

$= 16 + 12\sqrt{2} \times -\cos 45° + 4\sqrt{5}\cos\theta$

$= 16 - 12\sqrt{2} \times \frac{1}{\sqrt{2}} + 4\sqrt{5}\cos\theta$

$= 4 + 4\sqrt{5}\cos\theta$ ... ... (1)

$\angle BAE = \angle BAF - \angle EAF = \theta - 180°$

and $\angle DEA$= alt. $\angle BAE = \theta - 180°$

in rt. θd $\Delta\, AED$, $AE^2 = AD^2 + DE^2$

$= a^2 + \frac{a2}{4}$ a being the side of the square

$= \frac{5a2}{4}$

$\therefore$ AE $=\frac{a\sqrt{5}}{2}$

$\cos(\theta - 180°) = \frac{DE}{AE} = \frac{(\frac{a}{2})}{\frac{a\sqrt{5}}{2}} = \frac{1}{\sqrt{5}}$

i.e. $\cos(180° - \theta) = \frac{1}{\sqrt{5}} = -\cos\theta$

$\therefore \cos\theta = -\frac{1}{\sqrt{5}}$

From (1),

$R\cos\theta = 4 + 4\sqrt{5} \times -\frac{1}{\sqrt{5}} = 0$ ... ...(2)

$R\sin\theta = 12\sqrt{2}\,\sin 225° + 4\sqrt{5}\,\sin\theta + 20$

$= 12\sqrt{2}\,\sin(180° + 45°) + 4\sqrt{5}\,\sin\theta + 20$

$= -\,12\sqrt{2}\,\sin 45° + 4\sqrt{5}\,\sin\theta + 20$

$= -\,12\sqrt{2} \times \frac{1}{\sqrt{2}} + 4\sqrt{5}\,\sin\theta + 20$

$= 8 + 4\sqrt{5}\,\sin\theta$ ... ... (3)

From rt. $\angle$d $\Delta\, AED$,

$\sin(\theta - 180°) = \frac{AD}{AE} = \frac{a}{\frac{a\sqrt{5}}{2}} = \frac{2}{\sqrt{5}}$

i.e. $-\sin(180° - \theta) = \frac{2}{\sqrt{5}}$

$\therefore \sin\theta = -\frac{2}{\sqrt{5}}$

From (3),

$R\sin\theta = 8 + 4\sqrt{5} \times -\frac{2}{\sqrt{5}} = 8 - 8 = 0$ ... ...(4)

Squaring (2) and (4) and adding,

$R^2 = 0 + 0 = 0$

i.e $R = 0$

The forces are in equilibrium.

Check Your Progress

1. What is the resolved part?
2. What is the resolution of a forces into components?
3. What is the force-polygon?

4.9 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Resolution of force F can be resolved into two forces, which together produces the same effects that of force F. These forces are called the components of the force F.
2. Resolution of force F can be resolved into (or replaced by) two forces, which together produces the same effects that of force F. These forces are called the components of the force F. This process of replacing a force into its components is known as resolution of a force into components.
3. The required resultant is represented is magnitude and direction by the line AE. The same construction will apply for any number of forces. The figure ABCDE is said to be the force-polygon.

4.10 SUMMARY

- The algebraic sum of the resolved parts of two forces in any direction is equal to the resolved part of the resultant in the same direction.
- Two forces given in magnitude and direction have only one resultant, for we can construct only one parallelogram when two adjacent sides
- If any number of forces acting at a point are in equilibrium, the algebraic sums of the resolved parts of the forces in any two perpendicular directions must be zero separately.
- If the algebraic sum of the resolved parts of the forces acting at a point in any two perpendicular directions are zero separately, the forces will be in equilibrium.
- If G is the centroid on a ΔABC, forces represented by GA, GB, GC will be in equilibrium.
  i.e. $\overline{GA} + \overline{GB} + \overline{GC} = \overline{0}$
- The resolved part of a force in its own direction is the force itself.

4.11 KEY WORDS

- **Resolved part:** The algebraic sum of the resolved parts of a number of forces in any direction is equal to the resolved part of the resultant in the same direction.
- **Condition of equilibrium:** If any number of forces acting at a point are in equilibrium, the algebraic sums of the resolved parts of the forces in any two perpendicular directions must be zero separately.
- Conversely if the algebraic sum of the resolved parts of the forces acting at a point in any two perpendicular directions are zero separately, the forces will be in equilibrium.
- Two forces given in magnitude and direction have only one resultant, for we can construct only one parallelogram when two adjacent sides
- The resolved part of a force in its own direction is the force itself.

4.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. If a force P be resolved into two forces making angles of 45° and 15° with its direction, show that the latter force is $\frac{\sqrt{6}}{3}$P.

2. Find the components of force P along two directions making angle of 45° and 60° with P on opposite sides.

3. Two forces P and Q have a resultant R and the resolved part of R in the direction of P is of magnitude Q. Show that the angle between the forces in $2sin^{-1}\sqrt{\frac{P}{2Q}}$.

4. AD is an altitude of a triangle ABC. Show that the force AD acting along AD components $\frac{a^2+b^2-c^2}{2a^2}$AB and $\frac{c^2+a^2-b^2}{2a^2}$AC along AB and AC respectively.

5. If E, F are the feet of the perpendiculars from B and C upon the opposite sides of the triangle ABC, show that a forces P acting along EF can be replaced by P$\cos A$, P$\cos B$, P $\cos C$ acting along the sides of the triangle.

6. M is the point of trisection of the side AC of a ΔABC which is nearer to A. N is the point of trisection of the side AB which is nearer to B. Resolve a force represented in magnitude and direction by MN into three forces acting each along a side triangle.

7. ABCDE is a pentagon. Forces acting on a particle are represented in magnitude and direction by AB, BC, CD, 2DE, AD and AE. Find their resultant.

8. ABCD is a quadrilateral forces acting at a point are represented in magnitude and direction by AB, AD, BC and DC. Show that the resultant is along AC and its magnitude is 2AC.

9. The sides BC and DA of a quadrilateral ABCD are bisected in F and H respectively; show that if two forces parallel and equal to AB and DC act on a particle, then the resultant is parallel to HF and equal to 2 HF

10. Three forces P, Q, R in one plane act on a particle, the angles between R and Q, P and R, and P and Q being $\alpha, \beta\ and\ \gamma$ respectively. Show that their resultant is equal to

$$\sqrt{\{P^2 + Q^2 + R^2 + 2QR\cos\alpha + 2RP\cos\beta + 2PQ\cos\gamma\}}$$

11. If forces of magnitude P, Q, R act at a point parallel to the sides BC, CA, AB of a triangle ABC respectively, prove that the magnitude of their resultant is

$$\sqrt{\{P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C\}}$$

Hence deduce the triangle of forces.

12. Fifteen coplanar forces act at a point and are represented in magnitude and direction by the lines drawn from each of the vertices of a pentagon to the midpoints of those sides on which the vertex does not lie. Show that they are in equilibrium.
13. Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another.

14. Three forces act perpendicularly to the sides of a triangle at their midpoints and are proportional to the sides. Prove that they are in equilibrium.

4.13 FURTHER READINGS

Dr. M. K. Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.

Dr. M. K. Venkataraman, Dynamics, Agasthiar Publications, 13th Edition, 2009.

P. Duraipandian, Laxmi Duraipandian & Muthamizh Jayapragasam, Mechanics, S. Chand & Co. Pvt. Ltd, 2014.

BLOCK II
PARALLEL FORCES, COUPLES AND FRICTIONS

UNIT – V FORCES ACTING ON A RIGID BODY

STRUCTURE

5.0 Introduction
5.1 Objective
5.2 Parallel Forces
5.3 Resultant of Two Like and Unlike Parallel Forces
5.4 Resultant of a Number of Parallel Forces
5.5 Condition of Equilibrium of Three Coplanar
5.6 Centre of Two Parallel Forces
5.7 Moment of a Forces
5.8 Physical Significance of the moment of a Forces
5.9 Geometrical Representation of a Moment
5.10 Sign of the Moment
5.11 Unit of the Moment
5.12 Varigon's Theorem
5.13 Generalised Theorem of Moments
5.14 Worked Examples
5.15 Answer to Check Your Progress Questions
5.16 Summary
5.17 Keywords
5.18 Self Assessment questions and exericises
5.19 Further Reading

5.0 INTRODUCTION

In the previous unit we have considered the method of finding the resultant of two forces which meet at a point. We shall now consider how to find the resultant of two parallel forces. Such forces do not meet in a point and so we cannot find their resultant by direct application of the law of parallelogram of forces.

Two parallel forces are said to be *like* when they act in the same direction; they are said to be *unlike* when they act in opposite parallel directions.

5.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by parallel force.
- Discuss the theorems on forces acting on a rigid body.
- Describe moments.

5.2 PARALLEL FORCES

Let like parallel forces P and Q act at the points A and B of the rigid body respectively and let them be represented by the lines AD and BL. At A and B, introduce two equal and opposite forces F of arbitrary magnitude along the line AB and let them be represented be AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

The two forces F and P acting at the point A can be compounded into a single force R_1 represented by the diagonal AE of ADEG. Similarly the two forces F and Q acting at the point B will have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.

Produce EA and MB and let them meet at O. The two resultants R_1 and R_2 can considered to act at O. At O draw Y'OY || to AB and OX || to the directions of P and Q. Reresolve R_1 and R_2 at O into their original components.

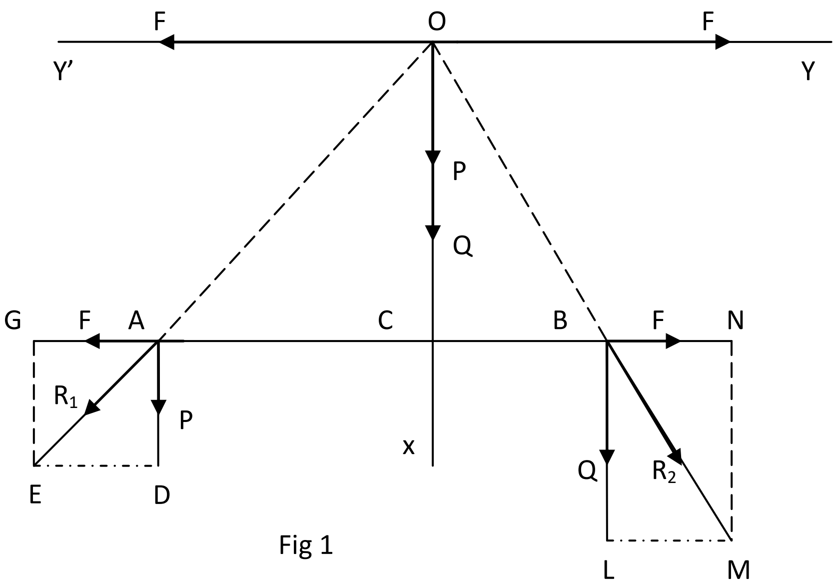


Fig 1

R_1 at O is equal to a force F along OY' and a force P along OX. R_2 at O is equal to a force F along OY and a force Q along OX. The two Fs at O cancel each other, being equal and opposite. We are now left with two forces P and Q acting OX. Hence their resultant is a force $(P + Q)$ acting along OX i.e. acting in a direction parallel to the original directions of P and Q.

Thus the magnitude of the resultant of two like parallel forces is their *sum*. The *direction* of the resultant is parallel to the components and in the same sense.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OAC and AED are similar.

$\therefore \frac{OC}{AD} = \frac{AC}{ED}$ i.e. $\frac{OC}{P} = \frac{AC}{F}$ or $F.OC = P.AC$ .....(1)

Triangles OCB and BLM are similar.

$\therefore \frac{OC}{BL} = \frac{CB}{LM}$ i.e. $\frac{OC}{Q} = \frac{CB}{F}$ or $F.OC = Q.CB$ .....(2)

From (1) and (2), we have $P.AC = Q.CB$

i.e. $\frac{AC}{CB} = \frac{Q}{P}$

i.e. the point C divides AB *internally* in the internally in the inverse ratio of the forces.

5.3 RESULTANT OF TWO LIKE AND UNLIKE PARALLEL FORCES

Let P and Q be two unequal and unlike parallel forces acting at the points A and B of the rigid body. Let $P > Q$ and let them be represented by AD and BL. At A and B, introduce two equal and opposite forces F of arbitrary magnitude along the line AB and let them be represented by AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

The two forces F and P acting at A can be compounded into a single force R_1 represented by the diagonal AE of the parallelogram AGED. Similarly the two forces F and Q acting at B have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.

Produce AE and MB and let them meet at O. The two resultants R_1 and R_2 can be considered to act at O. At, O draw Y'OY || to AB and OX parallel to direction of P and Q. Reresolve R_1 and R_2 at O into their original components. R_1 at O is equal to a force F along OY' and a force P along XO. R_2 at O is equal to a force F along OY and a force Q along OX. The Fs at O cancel each other, begin equal and opposite. We are now left with a force P along XO and a force Q along OX. Clearly the resultant is a force $P - Q$(as $P > Q$) acting along XO i.e. acting in a direction parallel to that of P.

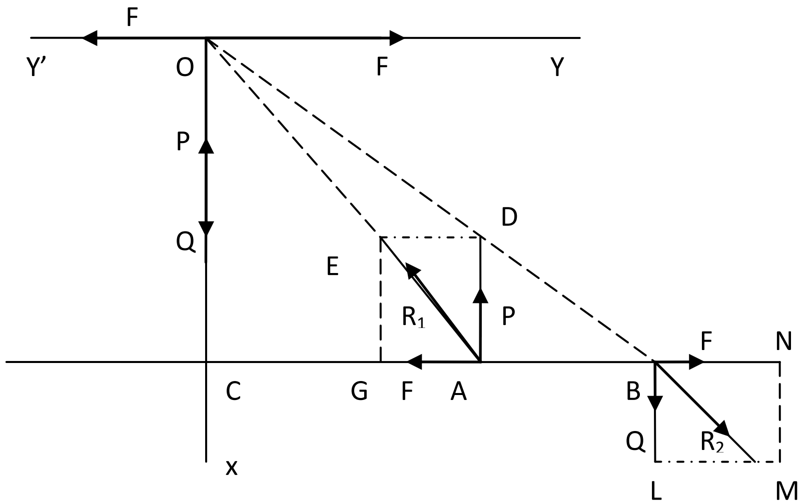


Fig. 2

Thus the *magnitude* of the resultant of two unlike parallel forces is their *difference*. The direction of the resultant is parallel to and the *sense of the greater component*.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OCA and EGA are similar.

$\therefore \frac{OC}{EG} = \frac{CA}{GA}$ i.e. $\frac{OC}{P} = \frac{CA}{F}$ or $F.OC = P.CA$ ..... (1)

Triangles OCB and BLM are similar.

$\therefore \frac{OC}{BL} = \frac{CB}{LM}$ i.e. $\frac{OC}{Q} = \frac{CB}{F}$ or $F.OC = Q.CB$ ..... (2)

From (1) and (2), we have $P.CA = Q.CB$

i.e. $\frac{CA}{CB} = \frac{Q}{P}$

i.e. The point C divides AB *externally* in the inverse ratio of the forces.

Note. As $P > Q$, CB must be $> CA$. Hence the resultant passes nearer the greater force.

Failure of the above construction:

NOTES

The construction for finding the resultant of two unlike parallel forces P and Q will fail, if $P = Q$ i.e. if the forces are equal in magnitude. In that case, in fig. 2, triangles AGE and BNM will be congruent.

$\angle GAE = \angle NBM$ and the lines AE and MB will be parallel.

There will be no such point as O.

Hence we conclude that *the effect of two equal and unlike parallel forces cannot be replaced by a single force.* Such pair of forces have no single resultant and they constitute what is called a couple, which will be considered later on.

5.4 NUMBER OF PARALLEL FORCES ACTING ON A RIGID BODY

If a number of parallel forces P, Q, R,… acting on a rigid body, their resultant can be found by successive applications of 5.3 & 5.4. First, we find the resultant R_1 of P and Q; then we find the resultant R_2 of R_1 and R and this process is continued, until the final resultant is obtained. If the parallel forces are all like, the magnitude of the final resultant is the sum of the forces. If the parallel forces are not all like, the magnitude of the resultant is the algebraic sum of the forces each taken with its proper sign.

5.5 CONDITIONS OF EQUILIBRIUM OF THREE COPLANAR PARALLEL FORCES

Let P, Q, R be three forces parallel in one plane and be in equilibrium. Draw a line to meet the lines of action of these forces at A, B and C respectively.

If all the three forces are in the same sense, equilibrium will be clearly impossible. Hence two of them (say P and Q) must be like and the third R unlike.

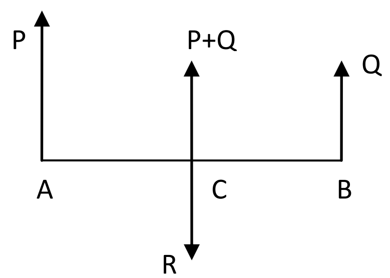


Fig. 3

The resultant of P and Q is $(P + Q)$, parallel to P or Q and hence, for equilibrium, R must be equal and opposite to $(P + Q)$.

$\therefore R = P + Q$ and the line of action of $P + Q$must pass through C.

$\therefore P.AC = Q.CB$

i.e. $\frac{P}{CB} = \frac{Q}{AC}$

and each $= \frac{P+Q}{CB+AC} = \frac{P+Q}{AB} = \frac{R}{AB}$

i.e. $\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$.

Thus, *if three parallel forces are in equilibrium, each is proportional to the distance between the other two.*

5.6 CENTRE OF TWO PARALLEL FORCES

Let P and Q be two parallel forces acting at two points A and B. Then, their resultant R passes through a point C, which divides AB internally or externally in the ratio $Q:P$.

i.e. $\frac{AC}{CB} = \frac{Q}{P}$ …….(1)

The position of C given by (1), depends only upon the positions of A and B and then magnitudes of the forces P and Q. It does not depends on the actual direction of P and Q. In other words, whatever be the common direction of parallelism of the forces P and Q, their resultant will always pass through a certain fixed point. This fixed point is called the centre of the fixed point. *Thus the centre of two parallel forces is a fixed point through which their resultant always passes whatever be the direction of parallelism.*

More generally, the resultant of a system of parallel forces of given magnitudes, acting at given points of a body, will always pass through a fixed point, for all directions of parallelism. This point is called *the centre of parallel forces*.

Example 1. *Two men, one stronger than the other, have to remove a block of stone weighing 300kgs. With a light pole whose length is 6 metre. The weaker man cannot carry more than 100kgs. Where must the stone be fastened to the pole, so as just to allow him his full share of weight?*

Let A be the weaker man bearing 100 kgs., his full share of the weight of the stone and B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that $AC = x$. Then the weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

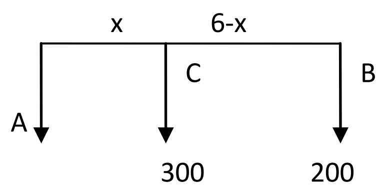


Fig. 4

$\therefore 100.AC = 200.BC$

i.e. $100x = 200\,(6 - x) = 1200 - 200x$

$\therefore 300x = 1200$ or $x = 4$.

Hence the stone must be fastened to the pole at the point distant 4 meters from the weaker man.

Example 2. *Two like parallel forces P and Q act on a rigid body at A and B respectively.*

a) *If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.*

b) *If P and Q interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d, where $d = \frac{P-Q}{P+Q}.AB$.*

(a) Let C be the centre of two parallel forces with P at A and Q at B.

Then $P.AC = Q.C$ ....(1)

If Q is changed to $\frac{P^2}{Q}$, (p remaining the same), let D be the new centre of parallel forces.

Then $P.AD = \frac{P^2}{Q}\ DB$ ....(2)

i.e. $PQ.AD = P^2.DB$

or $Q.AD = P.DB$ ....(3)

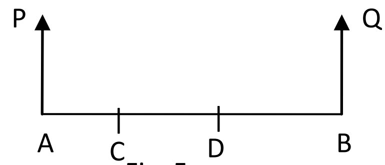


Fig. 5

Relation (3)shows that D is the centre of two like parallel forces, with Q and A and P at B.

(b) When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

$$CD = d.$$

From (3), $Q.(AC + CD) = P.(CB - CD)$

i.e. $Q.AC + Q.d = P.CB - P.d$

or $(Q + P).d = P.CB - Q.AC$

$$= P(AB - AC) - Q(AB - CB)$$

$$= P.AB - P.AC - Q.AB + Q.CB$$

$$= (P - Q).AB \quad [\because P.AC = Q.CB\ from\ (1)]$$

Or $d = \frac{P-Q}{P+Q.AB}$

Example 3. *Three like parallel forces, acting as the vertices of a triangle, have magnitudes proportional to the opposite sides. Show that their resultant passes through the incentre of the triangle.*

Let like parallel forces, P, Q, R act at A, B, C.

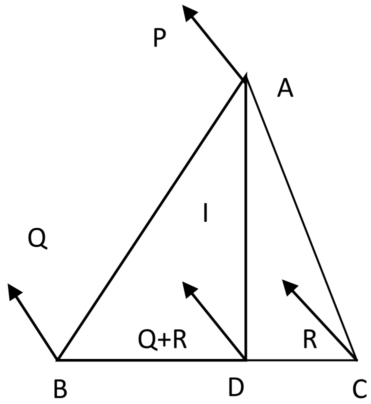


Fig. 6

It is given that $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$ …..(1)

Let the resultant of Q and R meet BC at D.

We know that the magnitude of the resultant is Q+R.

Also $\frac{BD}{DC} = \frac{force\ \ at\ C}{force\ \ at\ B} = \frac{R}{Q}$

$= \frac{c}{b}$ from (1)

$= \frac{AB}{AC}$

$\therefore AD$ is the internal bisector of ΔA.

We have now to find the resultant of the two like parallel forces, $Q + R$ at D and P at A.

Let this resultant meet AD at I.

Then $\frac{AI}{ID} = \frac{force\ \ at\ D}{force\ \ at\ B} = \frac{Q+R}{P}$

$= \frac{b+c}{a}$ from (1) ….(2)

From result (2), if it is clear that I is the incentre of the Δ.

[If I is the incentre of ΔABC and AD bisects $\angle A$ internally, we have the result $\frac{AI}{ID} = \frac{b+c}{a}$].

5.7 MOMENT OF A FORCES

When forces act on a particle, the only motion that can occur is a motion of translation. But a force acting on a rigid body may produce either a motion of translation or rotation or of translation and rotation combined. When there is a motion of translation alone, the force is measured by the product of the mass of the particle and the acceleration produced on it by the force. In the case of rotation, the idea of the *turning effect or moment* of a force is introduced.

Consider a sheet of cardboard pivoted freely at a fixed point O. If a force F acts along a straight line AB, it is clear that there will be no rotation if AB does not pass through O, the force will tend to rotate the sheet about O. This tendency to rotate the body will increase as the magnitude of the force increases and also as the perpendicular distance from O on the line of action of the force increases. Let ON be the length of the perpendicular from O on the line of action of F. The tendency to rotate varies as F when ON is constant. It also varies as ON when F is constant. Hence it varies as $F \times ON$ i.e. the product of F and ON, when both these quantities vary. This product is *called the moment of F about O. Thus the moment of a force about a point is defined to be the product of the force and the perpendicular distance of the point from the line of action of the force.*

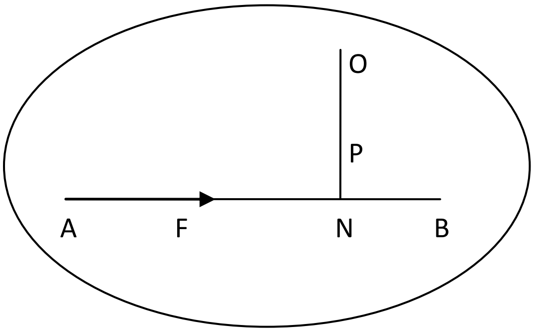


Fig. 7

5.8 PHYSICAL SIGNIFICANCE OF THE MOMENT OF A FORCE

The point F.ON will become zero only if either F is zero or ON is zero. When $ON = 0$, the point O is on the line of action of F. Hence if the moment of a force about a point is zero, either, (i) the force itself is zero, or

(ii) the line of action of the force passes through that point.

From the definition of the moment of a force about a point it is clear that it is a fitting measure for the turning effect of the force about the point. Thus the physical meaning for the moment of a force about a point is that it measure the tendency to rotate the body about that point.

5.9 GEOMETRICAL REPRESENTATION OF A MOMENT

Let a force F acting on a body be represented in magnitude, direction and line of action by the line AB.

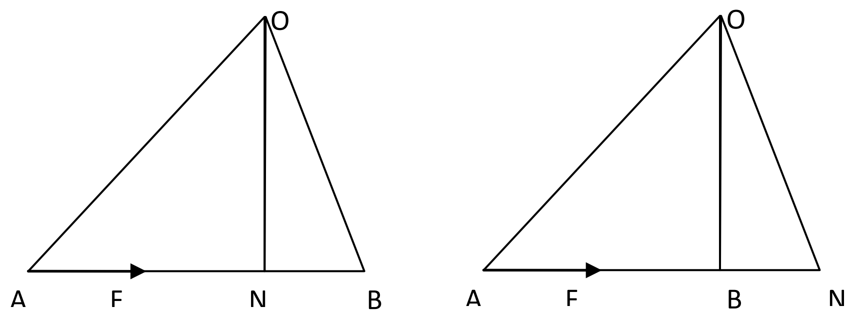


Fig. 8

Let O be any given point and ON the perpendicular from O on the AB or AB produced.

The moment of the force F about O $= F \times ON = AB \times ON = 2\Delta\, AOB$.

Hence *if a force is represented completely by a straight line, its moment about any point is given by twice the area of the triangle which the straight line subtends at that point.*

5.10 SIGN OF THE MOMENT

In fig., when the force F acts along AB, it will tend to the lamina in the anticlockwise direction i.e. in a direction opposite to that in which the hands of clock move. In such cases, the moment is said to be *positive*. If the force tends to turn the body in a clock wise direction, its moment is said to be *negative*.

Thus the moment, of a force about a point has both magnitude and direction and is therefore a vector quantity.

5.11 UNIT OF MOMENT

The moment of a unit force about a point at a unit perpendicular distance from the line of action of the force is defined as the unit for the measurement of moments. If the unit of force be a poundal and unit of distance be one foot the unit of moment is a poundal-foot. If the unit of force be a dyne and unit of distance be one centimeter, the unit of moment is a dyne-cm.

5.12 VARIGON'S THEOREM

The algebraic sum of the moment of two forces about any point in their plane is equal to the moment of their resultant about that point.

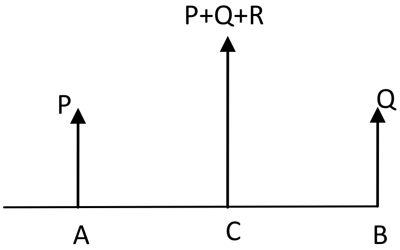


Fig. 9(a)

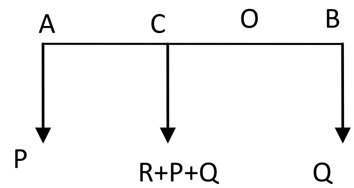


Fig. 9(b)

To prove this theorem, we consider two cases

Case I: Let the forces be parallel.

Let P and Q be two parallel forces and O any point in their plane. Draw AOB perpendicular to the forces to meet their lines of action in A and B.

The resultant of P and Q is a force $R(= P + Q)$ acting at C such that $P.AC = Q.CB$.

In fig 9(a),

The algebraic sum of the moments of P and Q about O

$= P.OA + Q.OB$

$= P(OC - AC) + Q(OC + CB)$

$= (P + Q).OC - P.AC + Q.CB$

$= (P + Q).OC \qquad [\because P.AC = Q.CB]$

$= R.OC =$ moment of R about O.

When the parallel forces P and Q are unlike and unequal, the theorem can be proved exactly in the same way.

Case II:

Let the forces meet at a point.

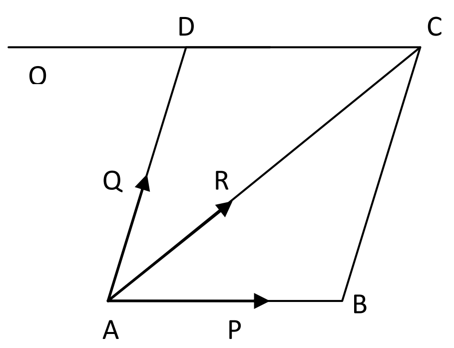


Fig. 10(a)

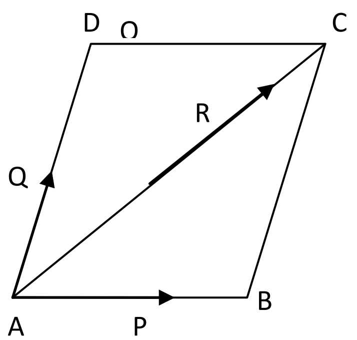


Fig.10(b)

Let the two forces P and Q act at A as shown in figs 10(a) and 10(b) and let O be any point in their plane. Through O, draw a line parallel to the direction of P meeting the line of action of Q at D. Choose the scale of representation such that length AD represents Q in magnitude. On the same scale, let length AB represent P. Complete the parallelogram BAD so that the diagonal AC represents the resultant R of P and Q.

In either figure, the moment of P, Q, R, about O are represented by $2\Delta AOB, 2\Delta AOD$ and $2\Delta AOC$ respectively.

In fig. 10(a), O lies outside the $\angle BAD$ and the moments of P and Q are both positive.

The algebraic sum of the moments of P and Q

$= 2\Delta AOB + 2\Delta AOD$

$= 2\Delta ACB + 2\Delta AOD$ [$\because \Delta AOB = \Delta ACB$]

$= 2\Delta ADC + 2\Delta AOD$ [$\because$ diagonal AC bisects the ||gm.]

$= 2(\Delta ADC + \Delta AOD)$

$= 2\Delta AOC$

= moment of R about O.

In fig. 10(b), O lies inside the angle BAD. The moment of P about O is positive while that of Q is negative.

The algebraic sum of the moments of P and Q

$= 2\Delta AOB - 2\Delta AOD$

$= 2\Delta ACB - 2\Delta AOD$

$= 2\Delta ADC - 2\Delta AOD$

$= 2(\Delta ADC - \Delta AOD)$

$= 2\Delta AOC$

= moment of R about O.

5.13 GENERALIZED THEOREM OF MOMENTS (PRINCIPLE OF MOMENTS)

If any number of coplanar forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Let $P_1, P_2, ...$ be any number of coplanar forces and O any point in their plane. Let R_1 be the resultant of P_1 and P_2, R_2 that of R_1 and P_3, R_3 that of R_2 and P_4 and so on until th final resultant R is obtained.

Applying Varignon's theorem to the forces P_1, P_2 and R_1, we have

moment of P_1 about O + moment of P_2 about O = moment of R_1 about O ….(1)

Similarly, applying the theorem to the forces R_1, P_3 and R_2, we have

moment of R_1 about O + moment of P_3 about O = moment of R_2 about O ….(2)

Combining (1) and (2), we have

moment of P_1 about O + moment of P_2 about O+ moment of P_3 about O = moment of R_2 about O. Proceeding thus, till all the forces are exhausted, we prove the above theorem.

Let $p_1, p_2, ...$ be the perpendicular distances of O from the lines of action of the forces $P_1, P_2, ...$ respectively and p the perpendicular distance of O from the line of action of the resultant R.

Then the above theorem can be written as

$P_1p_1 + P_2p_2 + \cdots = pR$

i.e. $\sum P_1p_1 = pR$ ...(1)

From this theorem, we derive the following important corollaries:

Corollary 1. If the basic point O about which moment is taken, happens to lie on the line of action of the resultant R, then $p = 0$.

$\therefore$ From (1), $\sum P_1p_1 = 0$.

Hence the algebraic sum of the moments of any number of coplanar forces about any point on the line of action of their resultant is zero.

Corollary 2. Suppose $\sum P_1p_1 = 0$.

Then from (1), $pR = 0$.

$\therefore$Either $p = 0$. It means that the basic point O about which moment is taken, lies on the line of action of the resultant.

If $R = 0$, it means there is no resultant for the system i.e. the forces are equilibrium.

Thus *if the algebraic sum of the moments of any number of forces about any point in their plane in zero, then either their resultant passes through the point about which moments are taken or the resultant is zero. In the latter case, the forces will be in equilibrium.*

Corollary 3. Suppose $R = 0$ i.e. the forces are in equilibrium.

Then form (1) $\sum \mathrm{P}_1\mathrm{p}_1 = p \times 0 = 0$.

Hence if a system of coplanar forces is in equilibrium, the algebraic sum of their moments about any point in their plane is zero.

This theorem enables us to find points on the line of action of a resultant of a system of forces. For, we have only to find a point about which the algebraic sum of the moments of the forces is zero and then the resultant must pass through that point.

5.14 WORKED EXAMPLES

Example 4. *Two men carry a load of $224\,kg.\ wt$, which hangs from a light pole of length 8m. Each end of which rests on a shoulder of one of the men. The point from which the load is hung is $2m$. Nearer to one man than the other. What is the pressure on each shoulder?*

AB is the light pole of length 8m. C is the point

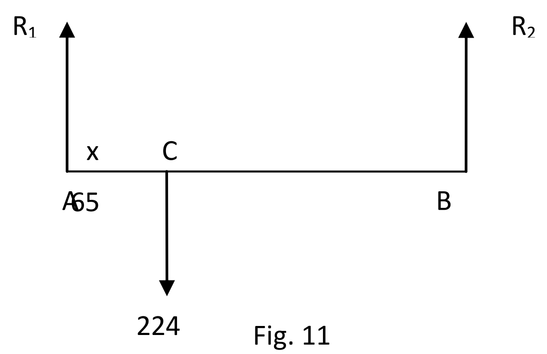


Fig. 11

from which the load of $224\ kgs$. is hung.

Let $AC = x$. Then $BC = 8 - x$.

It is given that $(8 - x) - x = 2$.

i.e. $8 - 2x = 2$ or $2x = 6$.

$\therefore x = 3$. i.e. $AC = 3$ and $BC = 5$.

Let the pressure at A and B be R_1 and $R_2\ kg.wt.$ respectively. Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1, R_2 and $224\ kg.wt.$ about any point must be equal to zero.

Taking moments about B,

$224CB - R_1.AB = 0$ (as the moment of R_2 about B is 0)

i.e. $224 \times 5 - R_1 \times 8 = 0$.

$\therefore R_1 = \frac{224 \times 5}{8} = 140$.

Taking moments about A,

$R_2\ AB - 224.AC = 0$.

i.e. $8R_2 - 224 \times 3 = 0$.

$\therefore R_2 = \frac{224 \times 3}{8} = 84$.

Note 1. For equilibrium, the weight of $224\ kgs$ must be equal and opposite to the resultant of R_1 and R_2.

$$\therefore R_1 + R_2 = 224.$$

Hence from this relation, we may find R_2 after finding R_1.

Note 2. In practice, instead of equating the algebraic sum of the moments of the forces about any point to zero, it will be convenient to equate the sum of the moments in one direction to the sum of the moments in the other direction. Hence in the above, taking moments about B, we have $R_1.AB = 224.BC$.

Example 5. *A uniform plank of length $2a$ and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that* $\frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{b}{a}$.

Let AB be the plank upon two vertical props at C and D. $CD = b$. The weight W of the plank acts at G, the midpoints of AB,

$AG = GB = a$

When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D then is zero.

There is upward reaction R_1 at C.

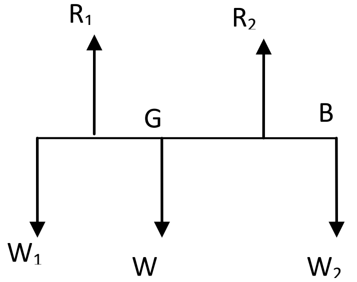


Fig. 12

Now, taking moments about C, we have

$W_1.AC = W.CG$

i.e.$W_1.(AG - CG) = W.CG$

or $W_1.AG = (W + W_1).CG$

i.e. $W_1.a = (W + W_1).CG$

or $CG = \frac{W_1.a}{W+W_1}$ ……(1)

When the weight W_2 is attached at B, there is loose contact at C. The reaction at C becomes zero. There is upward reaction R_2 about D.

Now taking moments about D, we get

Now, taking moments about C, we have

$W_2.BD = W.GD$

i.e.$W_2.(GB - GD) = W.CD$

or $GD(W + W_2) = W_2.GB = W_2.a$

or $GD = \frac{W_2.a}{W+W_2}$ ……(2)

But $CG + GD = CD = b$

$\therefore \frac{W_1.a}{W+W_1} + \frac{W_2.a}{W+W_2} = b$

Or $\frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{b}{a}$

Example 6. *The resultant of three forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC passes through the orthocenter. Show that the triangle must be obtuse angled. If* $\angle A = 120°$, *and* $B = C$, *show that* $Q + R = P\sqrt{3}$.

Let AD, BE and CF be the altitudes of the triangle intersecting at O, the orthocenter.

As the resultant passes through O, moment of the resultant about $O = 0$.

∴ Sum of the moments about P, Q, R about O is also =0.

Hence, taking moments about O, we have

$P.OD + Q.OE + R.OF = 0.$ …….(1)

In rt. $\angle d\Delta BOD, \angle OBD = \angle EBC = 90° - C$.

$$\therefore \tan(90° - C) = \frac{OD}{BD}$$

i.e. $\cot C = \frac{OD}{BD}$

or $OD = BD \cot C$ ……(2)

from rt. $\angle d\Delta ABD,$

$$\cos B = \frac{BD}{AB}$$

$$\therefore BD = AB.\cos B = c.\cos B$$

∴ From (2), $OD = c.\cos B\,.\cot C$

$$= c.\cos B\,.\frac{\cos C}{\sin C}$$

$$= \frac{c}{\sin C}.\cos B\ \cos C$$

$= 2R' \cos B\ \cos C$ ($\because\ \frac{c}{\sin\ C} = 2R', R'\ being\ the\ circumradius\ of\ the\ \Delta$)

Similarly, $OE = 2R' \cos C \cos A$

And $OF = 2R' \cos A \cos B$.

Hence (1) becomes

$$P, 2R' \cos B\ \cos C + Q.2R' \cos C \cos A + R.2R' \cos A\, cos\ B = 0.$$

Dividing by $2R' \cos A \cos B \cos C$, we get,

$\frac{P}{\cos\ A} + \frac{Q}{\cos\ B} + \frac{R}{\cos\ C} = 0$ ….(3)

Now, P, Q, R being magnitudes of the forces, are all positive of the terms must be negative.

Hence one of the cosines must be negative.

i.e. the triangle must be obtuse angled.

(Two of the cosines cannot be negative, as we cannot have two obtuse angles in the same triangle)

If $A = 120°$ and the other angles equal,

Then $B = C = 30°$.

Hence (3) becomes

$$\frac{P}{\cos 120°} + \frac{Q}{\cos 30°} + \frac{R}{\cos 30°} = 0$$

i.e. $\frac{P}{\left(-\frac{1}{2}\right)} + \frac{Q+R}{\left(\frac{\sqrt{3}}{2}\right)} = 0$

i.e. $P\sqrt{3} = Q + R$.

Check your Process

1. Define like and unlike forces.
2. Define centre of parallel forces.
3. State varigon's theorem.

5.15 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Two parallel forces are said to be **like** when they act in the same direction; they are said to be **unlike** when they act in opposite parallel directions.
2. **The centre of two parallel forces** is a fixed point through which their resultant always passes whatever be the direction of parallelism.
3. The algebraic sum of the moment of two forces about any point in their plane is equal to the moment of their resultant about that point.

5.16 SUMMARY

- Two parallel forces are said to be *like* when they act in the same direction; they are said to be *unlike* when they act in opposite parallel directions.
- Thus the magnitude of the resultant of two like parallel forces is their *sum*.
- The *direction* of the resultant is parallel to the components and in the same sense.
- *The effect of two equal and unlike parallel forces cannot be replaced by a single force.*
- *The centre of two parallel forces is a fixed point through which their resultant always passes whatever be the direction of parallelism.*

5.17 KEYWORDS

- **Like forces:** Two parallel forces are said to be *like* when they act in the same direction.
- **Unlike forces:** when two forces are acting in opposite parallel directions.

5.18 SELF ASSESSMENT QUESTIONS AND EXERICES

1. If the magnitudes of two unlike parallel forces P, Q, (P>Q) be increased by the same amount, show that the line of action of the resultant will move further off from P.
2. Three equal like parallel forces act at the middle points of the sides of a triangle; show that their resultant passes through the point of intersection of the medians of the triangle.
3. Four equal like parallel forces act at the corners of a square; show their resultant passes through the centre of the square.
4. A uniform plank ABC of length $12m$ and weight $80kgs$. rests on two supports A and B, one at the end A and the other at B, $4\frac{1}{2}m$, from the end C. A boy walks along the plank from A to C and just as he reaches C, the plank commences to tilt. Find the weight of the boy.
5. Show that any three forces acting along the sides of triangle cannot be in equilibrium.

5.19 FURTER READINGS

1. Dr. M.K Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.
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3. P. Duraipandian, Laxmi Duraipandian & Muthamizh Jayapragasam, Mechanics,S.Chand&Co.Pvt.Ltd,2014.

UNIT-VI COUPLES

STRUCTURE

6.0 Introduction
6.1 Objective
6.2 Couples
6.3 Equilibrium of Two Couples
6.4 Equivalence of Two Couples
6.5 Couples in Parallel Planes
6.6 Representation of a Couple by a Vector
6.7 Resultant of a Couple and a Plane
6.8 Answer to Check Your Progress Questions
6.9 Summary
6.10 Keywords
6.11 Self Assessment questions and exericises
6.12 Further Reading

6.0 INTRODUCTION

In the last unit we have seen that the general method of finding the resultant of two equal and unlike parallel forces fails i.e. the effects of two equal and unlike parallel forces cannot be replaced by a single force. A pair of such forces is called a couple.

6.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by Couple.
- Discuss the theorems on Equilibrium of two couples.
- Describe couples in parallel planes.

6.2 COUPLES

Definition. *Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.*

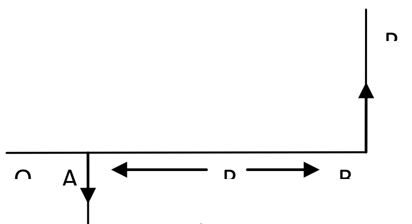

Fig. 1

Examples of a couple are the forces used in winding a clock or turning a tap. Such forces acting upon a rigid body can have only a rotatory effect on the body and they cannot produce a motion of translation.

Let P, p be the magnitudes of the forces forming a couple and O any point in their plane.

Draw OAB perpendicular to the forces to meet their lines of action in A and B.

The algebraic sum of the moments of the forces about O is

$= P.OB - P.OA$

$= P.(OB - OA) = P.AB$

And this value is independent of the position of O.

Thus the algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant and is equal to the product of either of the forces and the perpendicular distance between them. This algebraic sum measures the total turning effect of the couple upon the body and is called *the moment of the couple.*

Thus, the moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them.

The perpendicular distance $AB(= p)$ between the two equal forces P of a couple is called *the arm of the couple*. A couple each of whose forces is P and whose arm is p, as in fig. 1 is usually denoted by (P, p).

A couple is positive when its moment is positive i.e. if the forces of the couple tend to produce rotation in the anticlockwise direction and a couple is negative when the forces tend to produce rotation in the clockwise direction.

6.3 EQUILIBRIUM OF TWO COUPLES

Theorem.1. *If two couples, whose moments are equal and opposite, act in the same plane upon a rigid body, they balance one another.*

Let (P, p) and (Q, q) be two given couples such that $Pp = Qq$ in magnitude but opposite in sign.

Case 1: Let the forces P and Q be parallel.

Draw a straight line perpendicular to the lines of action of the forces, meeting them at A, B, C, D as in fig. 2.

Since the moments of the couples are equal, we have

$P.AB = Q.CD$ ….(1)

The downward like parallel forces P at A and Q at D can be compounded into a single force $P + Q$ acting at L such that

$P.AL = Q.DL$ ….(2)

(1)-(2) gives

$P.(AB - AL) = Q.(CD - DL)$

i.e. $P.B$L=Q.CL ….(3)

Result (3) shows that the resultant of the upward like parallel forces P at B and Q at C will also pass through L. The magnitude of this resultant is also $(P+Q)$ but it is opposite in direction to the previous resultant. Thus the two resultants balance each other. Hence the four forces forming the couples are in equilibrium.

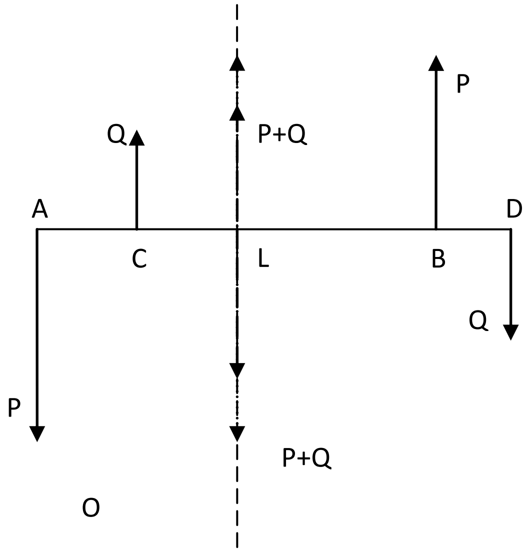

Fig. 2

Case 2: Let the forces P and Q intersect.

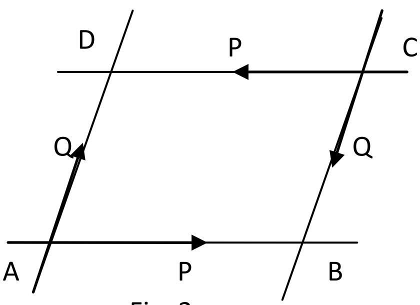

Fig. 3

Let the two forces P of the couple (P, p) meet the two forces Q of the couples (Q, q) at the points A, B, C, D. Clearly ABCD is a parallelogram.

Let AB represented P on some scale.

As the moments of the two couples are equal, we have

$P.p = Q.q$ .....(1)

Also $AB.p = AD.q$ (each being equal to the area of the || gm. ABCD)......(2)

(1) ÷ (2) gives

$\frac{P}{AB} = \frac{Q}{AD}$ .....(3)

(3) shows that the side AD will represent Q on the same scale in which the side AB represents P.

The two forces P and Q meeting at A can be compounded by ||gm. Law so that

$(P+Q)$ at $A = \overline{AB} + AD = AC$

Similarly $(P+Q)$ at C $= CD + CB = CA$.

Notes

The two resultants AC and CA being equal and opposite cancel each other.

Hence the four forces forming the couples are in equilibrium.

6.4 EQUIVALENCE OF TWO COUPLES

Theorem 2. *Two couples in the same plane whose moments are equal and of the same sign are equivalent to one another.*

Let (P, p) and (Q, q) be two couples in one plane having the same equal moments in magnitude and direction. Let (R, r) be a third couple, in the same plane, whose moment is equal to the moment of either (P, p) or (Q, q) only in magnitude but opposite in direction. By the previous theorem, the couple (R, r) will balance the couple (P, p). It will also balance the couple (Q, q). Hence the effects of the couples (P, p) and (Q, q) must be the same. In other words, they are equivalent.

This is a fundamental theorem on coplanar couples. Form this, it follows that a couple in a plane can be replaced by any other couple in the same plane, provided that the moment of the latter replacing couple is equal in magnitude and direction to the moment of the first couple. The only important criterion is that the moment of the new couple must be equal to that of the first couple in magnitude and sense.

Thus a couple (P, p) may be replaced by a couple $\left(F, \frac{Pp}{F}\right)$ in the same plane with its constituent forces each equal to F and the arm length begin equal to $\frac{Pp}{F}$. The moment of the couple is $= F\frac{Pp}{F} = P_p$ moment of the first couple. Also one force F may be taken to be acting in any line and direction, the other at the distance $\frac{Pp}{F}$ begin on that side so as to make the sign of the moment same as that of (P, p).

Similarly, the couple (P, p) may be replaced by a couple $\left(\frac{Pp}{x}, x\right)$ with a given arm x anywhere in the plane.

6.5 COUPLES IN PARALLEL PLANES

The effect of a couple upon a rigid body is not altered if it is transferred to a parallel plane provided its moment remains unchanged in magnitude and direction.

Consider a couple of forces P at the ends of arm AB in given plane. Let AL and BM be the line of action of the forces.

In any parallel plane, take a straight line CD equal and parallel toAB.

Then ABCD will be a parallelogram. The diagonals AD and BC will bisect each other, say at O.

At O, introduce two equal and opposite forces of magnitude 2P along EF, parallel to the forces P at A and B. By this, the effect of the given couple is not altered.

Now the unlike parallel forces P along AL and 2P along OE can be compounded into a single force P acting at D, since $\frac{AD}{OD} = \frac{2}{1} = \frac{2P}{P}$. This resultant force P acts along DN in the second plane. Similarly, the unlike parallel forces P along BM and 2P along OF can be compounded into a single force P acting at C along CK. We are therefore left with a couple of forces P at the ends of the arm CD in a plane parallel to that of the original couple.

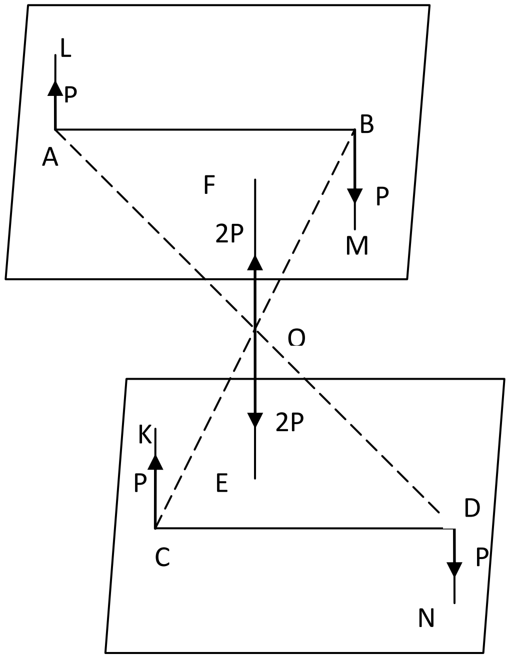

Fig. 4

Thus the given couple with the arm AB is equivalent to another couple of the same moment in a parallel plane, having its arm CD equal and parallel to AB. Now this couple with arm CD can be replaced in its own plane by another couple, provided the moment is unchanged in magnitude and direction as in 6.3. Hence we conclude that a couple in any plane can be replaced by another couple acting in a parallel plane, provided that the moments of the two couples are the same in magnitude and sign.

6.6 REPRESENTATION OF A COUPLE BY A VECTOR

From 6.3 and 6.4, it is clear that a couple is not localized in any particular plane, for it may be replaced by another couple of the same moment in the same plane or in any parallel plane. Thus the effect of a couple remains unaltered so long as its moment remains the same in magnitude and sense, whatever be the magnitude of its constituent forces, the length of its arm and its position in any one of a set of parallel planes in which it may be supposed to act.

A couple is therefore completely specified if we know (i) the direction of the set of parallel plane (ii) the magnitude of its moment (iii) the sense in which it acts. These three aspects of a couple can be conveniently represented by a straight line drawn (i) perpendicular to the set of parallel planes to indicate the direction (ii) of a measured length, to indicate the

Notes

moment of the couple and (iii) in a definite direction, to indicate the sense of the moment.

6.7 RESULTANT OF A COUPLE AND A PLANE

Theorem 3. *The resultant of any number of couples in the same plane on a rigid body is a single couple whose moment is equal to the algebraic sum of the moment s of the several couples.*

Let $(P_1, p_1), (P_2, p_2), (P_3, p_3)$ etc. be a number of couples acting in the same plane upon a body. Let AB represent the arm p_1 of the first couple (P_1, p_1) whose component forces P_1 act along AC and BD.

The moment of the second couple $(P_2, p_2) = P_2p_2$. This couple can be replaced by an equivalent couple, having its arm along AB and having its forces AC and BD.

If F is the forces of such a replacing couple,

We have $F.p_1 = P_2.p_2$.

$\therefore F = \frac{P_2p_2}{p_1}$

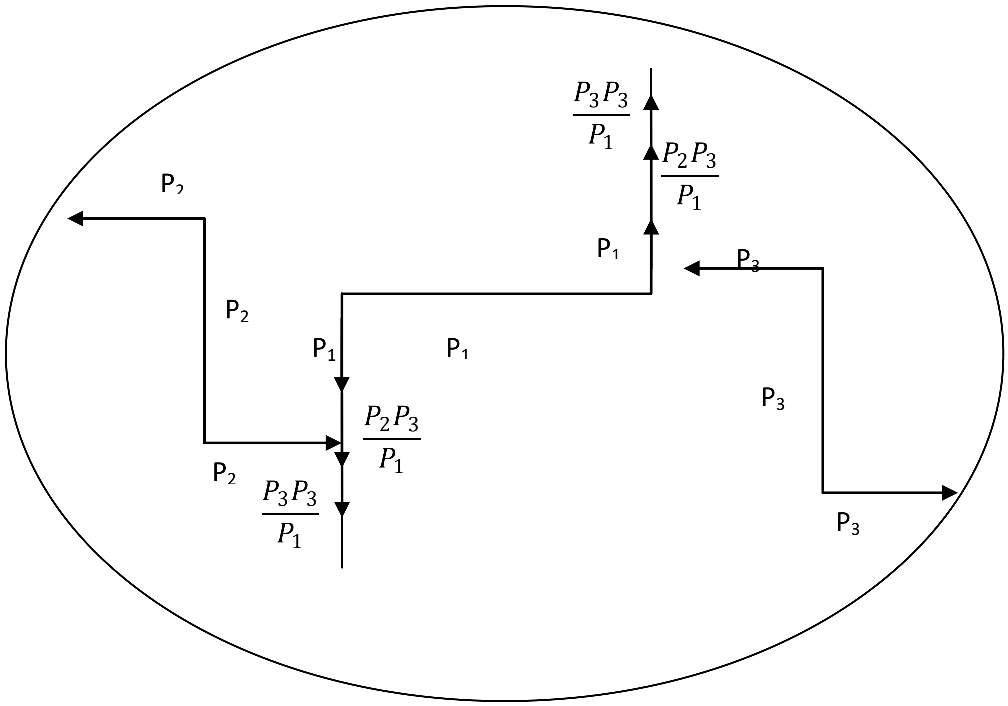

Fig. 5

Thus the couple P_2p_2 is replaced by another couple whose arm coincides with AB and whose component forces along AC and BD are magnitude $\frac{P_2p_2}{p_1}$.

Similarly the couple (P_3, p_3) is replaced by another couple $\left(\frac{P_3p_3}{p_1}, \mathrm{p}_1\right)$ with the forces $\frac{P_3p_3}{p_1}$ along AC and BD. This process is repeated for the other couples.

Finally, we get a single couple with the arm AB, each of whose component forces

$= P_1 + \frac{P_2p_2}{p_1} + \frac{P_3p_3}{p_1} + \ldots.$

The moment of this resultant couple

$$= \left(P_1 + \frac{P_2p_2}{p_1} + \frac{P_3p_3}{p_1} + \ldots.\right) \times p_1$$

$= P_1p_1 + P_2p_2 + P_3p_3 + \ldots.$

$=$ the algebraic sum of the moments of the several couples.

Note. (i) If all the component couples have not the same sign, we have merely to give each its proper sign and the same proof will apply.

(ii) If all the couples do not lie in the same plane but in different parallel planes, they can all be transferred into equivalent couples in one plane parallel to the given planes and then their resultant can be found.

Theorem 4. *A couple and a signal force acting on a body cannot be in equilibrium but they are equivalent to the single force acting at some other point parallel to its original direction.*

Let the given couple be (P, p) and the given force be F lying in the same plane. Let F act along AC.

Replace the couple (P, p) by another couple whose each force is equal to F. If x be the length of the arm of this new couple, its moment $= F.x = Pp$.

$$\therefore x = \frac{Pp}{F}$$

Place this couple such that one of its component forces F acts at A along the line of action of the given force F but in the opposite direction i.e. it acts along AD. The original force F along AC and the force F along AD balance. We are left with a force F acting at B parallel to AC, as the statical equivalent of the system.

Also $AB = x = \frac{Pp}{F}$

Hence the couple (P, p) and the force F are equivalent to an equal force F,

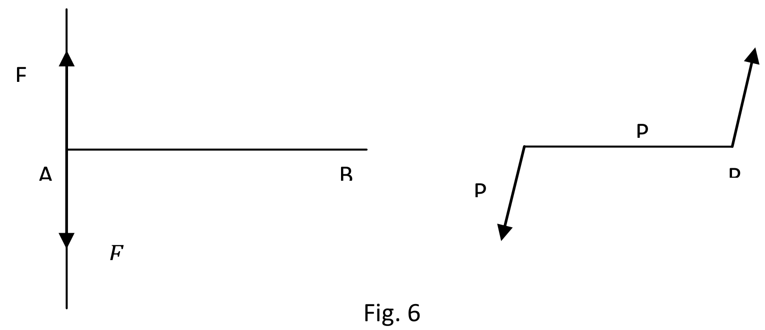

Fig. 6

parallel to its original direction, at a distance $\frac{Pp}{F}$ from its original line of action.

Theorem. 5. *A force acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body, together with a couple.*

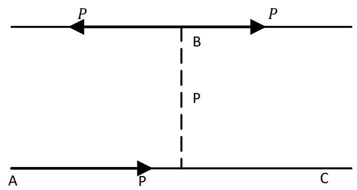

Fig. 7

Let P be a force acting at A along AC and B any arbitrary point. Let p be the distance of B from AC.

At B, apply two equal and opposite forces each equal and parallel to P along BL and BM. These two new forces being equal and opposite, will have no effect on the body. Of the three forces P along BM and P along AC from a couple and the remaining is the force P acting at B, parallel to the original force. Thus the statical equivalent of the original force P at A is an equal and parallel force P at B, together with a couple whose moment is Pp, where p is the perpendicular distance of B from AC.

Note. The moment of the couple is equal to the moment of the original force at A about B.

Theorem 6. *If there forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a triangle taken in order, they are equivalent to a couple whose moment is twice the area of the triangle.*

Let P, Q, R be three forces acting on a rigid body and represented in magnitude, direction and line of action by the sides BC, CA, AB of the triangle ABC taken in order. Through A draw LAM parallel to BC. At A, along AL and

AM introduce two equal and opposite forces, each equal to P. These two new forces, being equal and opposite, have no effect on the body.

Now the three forces P along AM, Q along CA, and R along AB act at the point A and they are completely represented by the sides of the ΔABC taken in order. Hence, by the triangle of forces, they are in equilibrium. We are left with a force P along AL and a force P along BC. These being two equal and opposite force form a couple whose moment

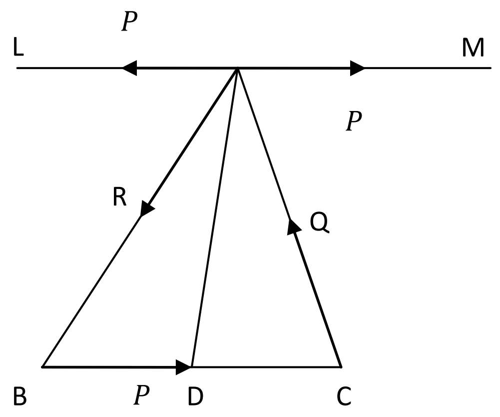

Fig. 8

$$= P.AD = BC.AD = 2\Delta ABC.$$

Theorem 7. If any number of forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a polygon taken in order, they are equivalent to a couple whose moment is twice the area of the polygon.

Let the forces be represented completely by the sides AB, BC, CD, DE, EF and FA of the closed polygon ABCDEF. Join AC, AD and AE.

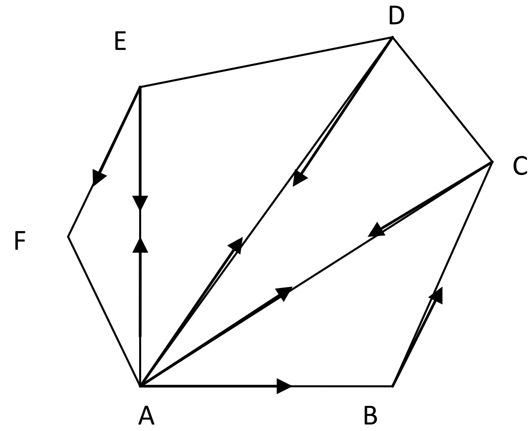

Fig. 9

Introduce along AC, AD and AE, pairs of equal and opposite forces represented completely by these lines. These new forces do not affect the resultant of the system.

Applying the theorem 6, we have

$\overline{AB}+\overline{BC}+\overline{CA}=$ a couple whose moment is equal to $2\Delta ABC$.

$\overline{AC}+\overline{CD}+\overline{DA}=$ a couple whose moment is equal to $2\Delta ACD$.

$\overline{AD}+\overline{DE}+\overline{EA}=$ a couple whose moment is equal to $2\Delta ADE$.

$\overline{AE}+\overline{EF}+\overline{FA}=$ a couple whose moment is equal to $2\Delta AEF$.

Adding vectorically,

$\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}+\overline{EF}+\overline{FA}=$ resultant of the four couples

= a single couple whose moment is equal to $2(\Delta ABC+\Delta ACD+\Delta ADE+\Delta AEF)$i.e. The resultant is a couple whose moment is equal to twice the area of the polygon ABCDEF.

Example 8. *ABC is an equilateral triangle of side a: D. E. F divide the sides BC, CA, AB respectively in the ratio 2:1. Three forces each equal to P act at D, E, F perpendicularly to the sides and outward from the triangle. Prove that they are equivalent to a couple of moment* $\frac{1}{2}Pa$.

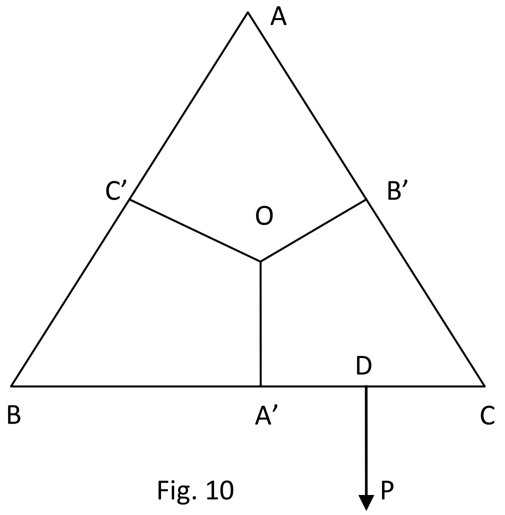

Fig. 10

Let O be the circumcentre (also the orthocentre) of the equilateral Δ and A', B', C' the middle points of the sides. OA' is $\perp$ to BC.

Applying theorem 5 , the force P acting at D $\perp$ to BC is equivalent to a parallel force P acting at O along OA' together with a couple whose moment

$$= P.AD = P.(A'C - DC) = P.\left(\frac{a}{2} - \frac{a}{3}\right) = \frac{Pa}{6}$$

Similarly, the force P acting at E $\perp$ to CA is replace by a parallel force P acting at O along OB' together with a couple whose moment $= \frac{Pa}{6}$.

The force P acting at F $\perp$ to AB is replaced by a parallel force P acting at O along OC' together with a couple whose moment $= \frac{Pa}{6}$.

The three equal forces P acting O $\perp$ to the sides of the triangle are in equilibrium by the perpendicular by the perpendicular triangle of forces.

The three couples having the same moment $\frac{Pa}{6}$ each in the same direction are equivalent to a single couple whose moment $= 3 \times \frac{Pa}{6} = \frac{Pa}{2}$.

Example 9. *Five equal forces ac along the sides AB, BC, CD, DE, EF of a regular hexagon. Find the sum of the moments of these forces about a point Q of AF at a distance x from A. Interpret the result and explain why it is so.*

Let a be the length of each side of the regular hexagon. Each interior angle of the regular hexagon $= 120°$.

We know that $AB||DE, BD||EF$ and $DC||AF, FB \perp BC, AE$ and DB are $\perp$ to AB.

Let equal force P act along the sides AB, BC, CD, DF and EF. Q is point on AF such that $AQ = x$.

Form Q, draw QL ⊥ to EA and QM ⊥ to BF.

Let AN be ⊥ to BF.

$$FB = FN + NB = a\cos 30° + a\cos 30° + 2a\cos 30°$$

$$AC = AE = BF = 2a\cos 30°$$

Moment of P along AB about Q

$= P.AL = P.x\cos 30°$ (from rt. $\angle d$ ΔAQL)

$= P.x\frac{\sqrt{3}}{2}$ …..(1)

Moment of P along BC about Q

$= P.MB = P.(FB - FM)$

$= P[2a\cos 30° - (a - x)\cos 30°]$

$= P[2a - a + x]\cos 30°$

$= P(a + x)\frac{\sqrt{3}}{2}$ ….(2)

Moment of P along CD about Q

=P. AC (∵ AF||CD and AC is ⊥ to CD)

$= P.2a\cos 30° = P.2a\frac{\sqrt{3}}{2} = Pa\sqrt{3}$ …..(3)

Moment of P along DE about Q

$= P.EL = P(AE - AL)$

$= P(2a\cos 30° - x\cos 30°)$

$= P(2a - x)\frac{\sqrt{3}}{2}$ …..(4)

Moment of P along EF about Q= $P.MF$

$= P(a - x)\cos 30°$

$= P(a - x)\frac{\sqrt{3}}{2}$ …..(5)

Adding up, the sum of the moments of the five forces about Q

$$= Px\frac{\sqrt{3}}{2} + P(a+x)\frac{\sqrt{3}}{2} + Pa\sqrt{3} + P(2a-x)\frac{\sqrt{3}}{2} + P(a-x)\frac{\sqrt{3}}{2}$$

$$= P\frac{\sqrt{3}}{2}(x + a + x + 2a - x + a - x)$$

$= P\frac{\sqrt{3}}{2}$ 6a $= 3Pa\sqrt{3} =$ a constant, independent of x.

The sum of the moments of the five forces about any point on the sixth side AF is constant.

Introduce two equal and opposite forces, each equal to P along the sixth side. These new forces do not affect the resultant of the system. We have now seven forces. The moment of the new force P introduced along AF about Q is =0.

The other six forces act along the sides of the hexagon and are represented in magnitude, direction and line of action by the sides of the hexagon.

Hence by theorem 6.6, they are equivalent to a couple whose moment is $= 2 \times$ area of the hexagon$= 2 \times 6 \times a^2 \frac{\sqrt{3}}{4}$

$= 3a^2\sqrt{3} = 3a\sqrt{3}P$ (as P is represented in magnitude by a).

Check your Process

1. Define Couple.
2. What is meant by moment of the couple.

6.8 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Two equal and unlike parallel forces not acting at the same point are said to constitute a **couple.**
2. Thus, the **moment of a couple** is the product of either of the two forces of the couple and the perpendicular distance between them.

6.9 SUMMARY

- Two equal and unlike parallel forces not acting at the same point are said to constitute a *couple.*
- *Thus, the moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them.*
- The perpendicular distance $AB(= p)$ between the two equal forces P of a couple is called *the arm of the couple*.

6.10 KEYWORDS

- **Couple:** *Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.*

Notes

- **Unlike forces:** *Thus, the moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them*

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6.11 SELF ASSESSMENT QUESTIONS AND EXERICES

1. Forces of magnitudes $1,2,3,4,2\sqrt{2}$ act respectively along the sides AB, BC, CD, DA and the diagonal AC of the square ABCD. Show that their resultant is a couple and find its moment.
2. Forces of $3,4,5,6$ and $2\sqrt{2}$ act respectively along the sides AB, BC, CD and DA and along the diagonal AC of the square ABCD. Find the resultant.

6.12 FURTER READINGS

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UNIT-VII EQUILIBRIUM OF THREE FORCES ACTING ON A RIGID BODY

STRUCTURE

7.0 Introduction
7.1 Objective
7.2 Three Forces Acting on As Rigid Body
7.3 Three Coplanar Forces
7.4 Conditions of Equilibrium
7.5 Two Trigonometrical Theorem and Simple Problems
7.6 Answer to Check Your Progress Questions
7.7 Summary
7.8 Keywords
7.9 Self Assessment questions and exericises
7.10 Further Reading

7.0 INTRODUCTION

There is a large class of problems in which a body is in equilibrium under the action of three forces. We shall first prove that, *if three forces acting on a rigid body are in equilibrium, they must be coplanar.*

7.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by parallel force.
- Discuss the theorems on forces acting on a rigid body.
- Describe moments.

7.2 THREE FORCES ACTING ON AS RIGID BODY

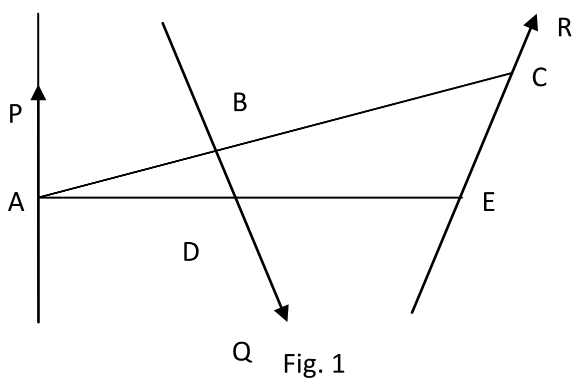

Fig. 1

Let P, Q, R be the three forces in equilibrium. Take any point A on the line of action of P and any point B on the line of action of Q, such that AB is not parallel to R. Then, the three forces being in equilibrium, the sum of their moments about the line AB is zero. But P and Q intersect AB and therefore their moments about AB are each zero. Hence the moment of R about AB is zero.

$\therefore$ *R* is either parallel to AB or R intersects AB.

But we have chosen the points A and B such that R is not parallel to AB.

$\therefore$ *R* must intersect AB at a point, say C.

Similarly, if D is some other point on Q such that AD is not || to R, we can prove that R must intersect AD also at a point, say E.

Since the line BC and DE intersect at A, BD and CE must lie in one plane and A is on this plane.

i.e. A is a point on the plane formed by Q and R.

But A is any point on the line of action of P.

$\therefore$Every point on P is a point on the plane formed by Q and R.

i.e. P, Q, R are in one plane.

7.3 THREE COPLANAR FORCES, CONDITIONS EQUILIBRIUM

If three coplanar forces acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.

Let P, Q, R be three coplanar forces acting on a body and keep it in equilibrium.

Then R must be equal and opposite to the resultant of P and Q.

Now, P and Q being coplanar must either be parallel or intersect.

Case 1: If P and Q are parallel (like or unlike), their resultant is also a parallel force. As R balances the above resultant, it must act in the same line but in opposite direction. So R also is in the same direction as that of P and Q.

i.e. P, Q, R are all parallel to one another.

Case 2: Let P and Q meet at a point O. Then, by parallelogram law, their resultant is a force through O. As this is balanced by the third force R, the line of action of R must also pass through O.

i.e. the three forces are concurrent.

Important Note. In the above discussion, P and Q can never form a couple, since we know that a couple and a force can never be in equilibrium.

7.4 CONDITIONS OF EQUILIBRIUM

When the number of forces acting on a rigid body in equilibrium is three and when the forces are not parallel, we can use the methods which apply to forces acting on a particle. Thus we can use Lami's theorem, or the triangle of forces or we can resolve the forces in two directions at right angles to each other.

When the three forces in equilibrium are parallel, we use the condition that each is proportional to the distance between the other two.

In all cases, it is important to draw a figure with the three forces clearly shown, either all parallel or meeting in a point.

7.5 PROCEDURE TO BE FOLLOWED IN SOLVING ANY STATICAL PROBLEM

In solving any statical problem the student should proceed in the following manner:

1) First draw the figure according to the conditions given.
2) Mark all the forces acting on the body or bodies, bearing in mind the following fundamental points:
i) The weight of a body acts vertically downwards through its centre of gravity.
ii) When a body is leaning against a smooth surface, the reaction on the body is normal to the surface.
iii) When a rod is resting on a smooth peg, the reaction of the peg on the rod is perpendicular to the rod.
iv) The tension in a light string is the same throughout its length and this tension is unaffected by the string passing over smooth pegs or pulleys. If the pulley is rough, the tension is different on the two sides of the pulley.
v) The resultant of two equal forces bisects the angle between them.
3) In addition to the above considerations, we can use the fact there are only three non-parallel forces, they must meet in a point. Thus, if three forces are in equilibrium and two of them meet at a point O, the third also must pass through O. This consideration will enable us to draw an accurate figure showing the position of the body.

7.6 TWO TRIGONOMETRICAL THEOREM AND SIMPLE PROBLEMS

The following two important trigonometrical theorems will be found to be highly useful in the solution of many statical problems:

If D is any point on the base BC of triangle ABC such that $\frac{BD}{DC} = \frac{m}{n}$ *and* $\angle ADC = \theta, \angle BAD = \alpha$ *and* $\angle DAC = \beta$ *then*

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta \qquad ....(1)$$

and $(m+n)\cot\theta = n\cot B - m\cot C$ ....*(2)*

Proof.

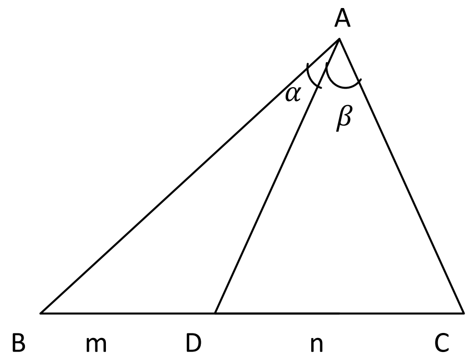

Fig. 2

(1) $\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{DA}.\frac{DA}{DC}$

$= \frac{\sin\angle BAD}{\sin\angle ABD}.\frac{\sin\angle ACD}{\sin\angle DAC}$

$= \frac{\sin\alpha}{\sin(\theta-\alpha)}.\frac{\sin(\theta+\beta)}{\sin\beta}$

$[\because \angle ACD = 180° - (\theta+\beta)]$

$= \frac{\sin\alpha(\sin\theta\cos\beta+\cos\theta\sin\beta)}{\sin\beta(\sin\theta\cos\alpha+\cos\theta\sin\alpha)}$

$= \frac{\cot\beta+\cot\theta}{\cot\alpha-\cot\theta}$ (dividing the numerator and denominator by $\sin\alpha\sin\beta\sin\theta$)

$$n(\cot\beta + \cot\theta) = m(\cot\alpha - \cot\theta)$$

Or $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(2) Again, $\frac{m}{n} = \frac{\sin\angle BAD}{\sin\angle ABD}.\frac{\sin\angle ACD}{\sin\angle DAC}$

$= \frac{\sin(\theta-B).\sin C}{\sin B.\sin(C+\theta)}$ $[\because \angle DAC = 180° - \overline{\theta + C}]$

$= \frac{\sin C(\sin\theta\cos B-\cos\theta\sin B)}{\sin B(\sin C\cos\theta+\cos C\sin\theta)}$

$= \frac{\cot B+\cot\theta}{\cot\theta-\cot C}$ (dividing the numerator and denominator by $\sin B\sin C\sin\theta$)

i.e. $m(\cot\theta + \cot C) = n(\cot B - \cot\theta)$

Or $(m+n)\cot\theta = n\cot B - m\cot C$.

7.7 SOME ARTIFICES

In some problems on equilibrium of a body acted on by three forces, we may require a relation between the geometrical quantities which define the equilibrium. In such cases, we may use one or more of the following methods:

i. Use one of the well-known trigonometrical theorems of 7.4 .
ii. Take moments of the forces about a suitable point.
iii. Draw, from a suitable point, a perpendicular on the vertical through the point of intersection of the forces and then calculate the length of this perpendicular in two or more ways.

In problem requiring the magnitudes of the forces acting, we can use Lami's theorem.

The procedure is illustrated in the following worked examples.

Example 1. *A uniform rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length l, tied to one end of the rod, the other end of the string being attached to a point in the wall: show that the rod can rest inclined to the wall at an angle θ given by*

$$cos^2\,\theta = \frac{l^2 - a^2}{3a^2}.$$

What are the limits of the ratio of $a: l$ in order that equilibrium may be possible?

AB is the rod of length a, with G its centre of gravity and BC is the string of length l.

The forces acting on the rod are:

i. Its weight W acting vertically downwards through G.
ii. The reaction R_A at A which is normal to the wall and therefore horizontal.
iii. The tension T of the string along BC.

These three forces in equilibrium not being all parallel, must meet in a point L, as shown in the figure.

Let the string make an angle α with the vertical.

$$\therefore \angle ACB = \alpha = \angle GLB$$

Also $\angle LGB = 180° - \theta$ and $\angle ALG = 90°$

Using the first trignonometrical theorem of sec.5 to ΔALB and noting that $AG{:}GB = 1{:}1$, we have

$(1+1)\cot(180° - \theta) = 1.\cot 90° - 1.\cot\alpha$

i.e. $-2\cot\theta = -\cot\alpha$

or $\cot\theta = \cot\alpha$ ……….(1)

Draw $BD \perp$ to CA.

Form rt. $\angle d\,\Delta CDB, BD = BC.\sin\alpha = l.\sin\alpha$ and from rt. $\angle d\,\Delta ABD, BD = AB.\sin\theta\ \ a.\sin\theta$.

$\therefore l.\sin\alpha = a.\sin\theta$ ....(2)

Eliminate α between (1) and (2).

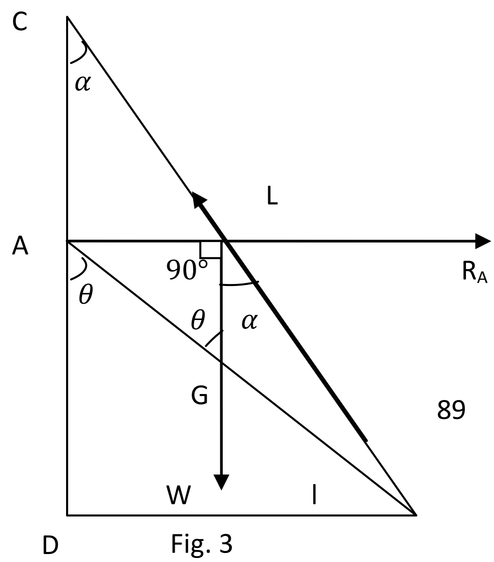

Fig. 3

We know that $Cosec^2\alpha = 1 + \cot^2\alpha$ …………(3)

Form (2), $\sin\alpha = \frac{a\sin\theta}{l}$

$\therefore \operatorname{cosec}\alpha = \frac{l}{a\sin\theta}$ ………………(4)

Substituting (4) and (1) in (3), we have

$\frac{l^2}{a^2\sin^2\theta} = 1 + 4\cot^2\theta$

i.e.

$\frac{l^2}{a^2} = \sin^2\theta + 4\cos^2\theta = 1 + 3\cos^2\theta$

$\therefore 3\cos^2\theta = \frac{l^2}{a^2} - 1 = \frac{l^2 - a^2}{a^2}$

$\therefore \cos^2\theta = \frac{l^2 - a^2}{3a^2}$ …………(5)

For the above equilibrium position to be possible, $cos^2\theta$ must be positive and less than 1

$\therefore l^2 - a^2 > 0$ i.e. $l^2 > a^2$.

Also $\frac{l^2 - a^2}{3a^2} < 1$ i.e. $l^2 - a^2 < 3a^2$ or $l^2 < 4a^2$ i.e. $a^2 > \frac{l^2}{4}$

$\therefore a^2$ lies between $\frac{l^2}{4}$ and l^2.

$\therefore \frac{a^2}{l^2}$ lies between $\frac{1}{4}$ and 1.

Or must lie between $\frac{1}{2}$ and 1.

Check your Process

1. Define equilibrium .
2. What will happen when three coplanar forces acting in a rigid body.
3. State two trigonometrical theorem.

7.8 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. If three forces acting on a rigid body are in **equilibrium**, they must be coplanar.
2. If **three coplanar forces** acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.
3. If D is any point on the base BC of triangle ABC such that $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta, \angle BAD = \alpha$ and $\angle DAC = \beta$ then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$ and $(m+n)\cot\theta = n\cot B - m\cot C$.

7.9 SUMMARY

- *If three forces acting on a rigid body are in equilibrium, they must be coplanar.*
- *If three coplanar forces acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.*

7.10 KEYWORDS

- **Equilibrium:** *if three forces acting on a rigid body are in equilibrium, they must be coplanar.*
- **Three coplanar forces***: If three coplanar forces acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.*

7.11 SELF ASSESSMENT QUESTIONS AND EXERICES

1. A uniform rod has its lower end fixed to a hinge and its other end attached to a string which is tied to a point vertically above the hinge. Show that the direction of the action at the hinge bisects the string.
2. A uniform rod can turn freely about one of its ends and is pulled aside from the vertical by a horizontal force acting at the other end of the rod, equal to half its weight. Prove that the rod will rest at an inclination of 45° to the vertical.
3. A heavy uniform rod of length 2a lies over a smooth peg with one end resting on a smooth vertical wall. If c is the distance of the peg from the wall and θ the inclination of the rod to the wall, show that $c = a\sin^3\theta$.

7.12 FURTER READINGS

- Dr. M.K Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.
- Dr. M.K Venkataraman, Dyanamics, Agasthiar Publications, 13th Edition, 2009.
- P. Duraipandian, Laxmi Duraipandian & Muthamizh Jayapragasam, Mechanics,S.Chand&Co.Pvt.Ltd,2014.

Notes

UNIT-VIII FRICTION

STRUCTURE

8.0 Introduction
8.1 Objective
8.2 Experimental Results
8.3 Statical, Dynamical and Limiting Friction
8.4 Laws of Friction
8.5 Friction
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8.0 INTRODUCTION

Till now, we have been studying the problems involving equilibrium of smooth bodies. When two smooth bodies are in contact with each other, the mutual action between them is entirely along the common normal at the point of contact. There is no force in the tangential direction. Hence there is no force tending to prevent one smooth body from sliding over the other.

But practically, there are no bodies which are perfectly smooth. All bodies are rough to some extent. Thus if we attempt to drag a heavy body along the ground by means of a horizontal force, a resistance is felt to the motion of the body. This resistance is due to the roughness of the ground and is called the normal friction. Thus, in the case of rough bodies in contact, besides the normal reaction, a tangential reaction i.e. a force acting in a direction perpendicular to the normal reaction is called into play. This tangential force between two bodies in contact prevents the one from sliding over the other. Such a force is called the force of friction.

8.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by friction.
- Discuss about the different types of friction.
- Describe laws of friction.

Definition. *If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their* *point of contact to prevent one body from sliding on the other, is called friction; the force exerted is called the force is friction.*

8.2 EXPERIMENTAL RESULTS

Suppose a heavy body is placed on a table and is pulled in a horizontal direction by a force P. It is found that, up to a certain value of P, the body does not move. The normal reaction R of the table and the weight W of the body are acting in the vertical direction and so, they have no effect in the horizontal direction. They are not responsible for stopping the motion of the body. Since the body is at rest, there must be some force in the horizontal direction to oppose the force P. This force F is the force of friction between the body and the table.

As P is gradually increased, the force F also increases so as to balance P at each instant. This state will continue till P attains a certain value when the body is just on the point of motion. At the stage, the force of friction has attained its maximum value and equilibrium is about to be broken. When P is further increased, F cannot increase further, since it has already reached its maximum. The equilibrium is actually broken and the body begins to move.

Thus we find that so long as the body remains at rest, the force of friction depends on P and is just sufficient to resist P. In this case, friction is called statical friction. Thus statical friction is a self-adjusting force and is just sufficient to maintain equilibrium. If P ceases to exist, F also vanishes, as otherwise, the body will move in the opposite direction. Also the amount of statical friction varies from zero up to a maximum value.

8.3 STATICAL, DYANAMIC AND LIMITING FRICTION

When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called statical friction.

When one body is just on the point of sliding on another, the friction attains its maximum value and is called limiting friction; the equilibrium in this case is said to be limiting.

When motion ensures by one body sliding over another, the friction exerted is called dynamical friction.

8.4 LAWS OF FRICTION

Friction is not a mathematical concept; it is a physical reality. The results of physical observation and experiment are formulated as the Law of Friction.

Law 1. When two bodies are is contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.

Law 2. When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

Law 3. The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.

Law 4. The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.

Law 5. (Law of dynamical friction)

When motion ensures by one body sliding over the other, the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

Note: These laws are experimental, and cannot be accepted as rigorously accurate but they express fairly accurately the results of a large number of experiments.

8.5 FRICTION

A passive force: It should be noted that friction is only a resisting force and appears only when necessary to prevent or oppose the motion of the point of contact. It cannot by itself produce motion of a body but it maintains relative equilibrium. It is a self-adjusting force. It assumes such magnitude and direction as to balance other forces acting on the body. Such a type of force is called a *passive force*. Friction is thus a purely passive force:

The force of friction, through considered to be dissipative is really beneficent, for, without it, most forms of motion would be impossible. If there were no friction of the ground, walking would have been impossible. Screws or nails would not stick to wood. Wheels and carriages would not roll. Thus friction is indirectly the agent for producing motion, though often it is recognized as a waste of energy and a source of loss.

8.6 COEFFICIENT OF FRICTION

In 8.4, by law 3, we know that limiting friction between two bodies bears a constant ratio to the normal reaction between them. *The ratio of the limiting friction to the normal reaction is called the coefficient of friction.* It is usually denoted by the letter μ.

Let F be the friction and R the normal reaction between two bodies when equilibrium is limiting.

Then $\frac{F}{R} = \mu, i.e. F = \mu R.$

The constant μ depends on the nature of the materials in contact. It is different for different pairs of substance and is ordinarily less than unity.

Since friction is maximum when it is limiting, μR is the maximum value of friction. When equilibrium is non limiting, F is less than $\mu\ R$ and $\frac{F}{R} < \mu$.

Note. 1. In limiting equilibrium, F and R may both vary with the masses of the bodies even for the same pair of substances but the ratio $\frac{F}{R}$ is constant being equal to μ.

3. We must not assume that friction is always equal to μR. It has this value only when motion is about to take place. Otherwise, it may have any value from zero upto μR.

8.7 ANGLE OF FRICTION

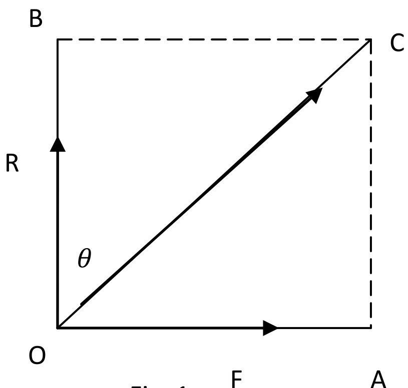


Fig. 1

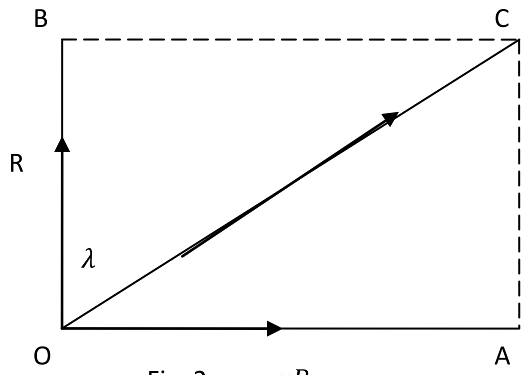


Fig. 2

Suppose one body is kept in equilibrium by friction on another. At the point of contract Q, two forces act on the first body, namely the normal reaction and the force of friction. These two act in perpendicular directions and they can be compounded into a single force. This single force is called the resultant reaction or the total reaction.

In fig. 1, let $\overline{OA} = F$, the force of friction and $\overline{OB} = R$ the normal reaction. Let $\overline{OC}$ be the resultant of F and R.

It $\angle BOC = \theta, \tan\theta = \frac{BC}{OB} = \frac{OA}{OB} = \frac{F}{R}$ ....(1)

As F increases from 0, the value of θ increases until the friction F reaches its maximum value. In that case, the equilibrium is limiting and the angle made by the resultant reaction with the normal is called the *angle of friction* and it is denoted by λ.

Hence the greatest value of θ is λ.

When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called the angle of friction and it is called the angle of friction and it is denoted by λ.

In fig. 2, $\overline{OA}$ represents the limiting friction which $= \mu R, \mu$ begin the coefficient of friction.

$\overline{OC}$ is the resultant of $\overline{OA}$ and $\overline{OB}$.

$\angle BOC = \lambda$ =the angle of friction.

$\tan\lambda = \frac{BC}{OB} = \frac{OA}{OB} = \frac{\mu R}{R} = \mu.$

i.e. The coefficient of friction is equal to the tangent of the angle of friction.

8.8 CONE OF FRICTION

From 8.7 we see that the greatest angle which the direction of the resultant reaction can make with the normal is λ i.e. $tan^{-1}(\mu)$.

Now the motion of one body at O, its point of contact with another, can take place in any direction perpendicular to the normal. Hence when two bodies are in contact, we can consider a cone drawn with the point of contact as the vertex, the common normal as the axis and its semi-vertical angle being equal to λ,the angle to friction. It is clear that the resultant reaction will have a direction which entirely lies within the surface or on the surface of that cone. It cannot fall outside the cone such a cone is called the cone of friction.

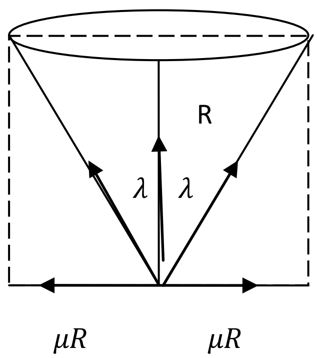


Fig. 3

8.9 EQUILIBRIUM OF A PARTICLE ON A ROUGH INCLINED PLANE

Let a particle of weight W be placed at A on a rough inclined plane, whose inclination to the horizontal is . The forces acting on it are:

(i) its weight W acting vertically downwards

(ii) the frictional force F acting along the inclined plane upwards. (If there had been no friction, the body would have tendency to move downwards. Hence friction will act upwards).

(iii) the normal reaction R, perpendicular to the plane.

Resolving along and perpendicular to the plane, we get

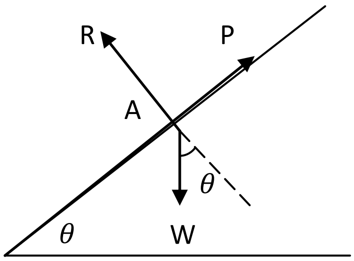


Fig. 4

$F = W \sin\theta$ ....(1)

And $R = W\cos\theta$ ....(2)

$\therefore \frac{F}{R} = \tan\theta$ ....(3)

We know that $\frac{F}{R}$ is always $< \mu$.

Hence for equilibrium, $\tan\theta < \mu$.

i.e. $\tan\theta < \tan\lambda$, λ being the angle of friction or $\theta < \lambda$.

Suppose θ, the inclination of the plane, is gradually increased. When $\theta = \lambda$, then $\frac{F}{R} = \tan\lambda = \mu$.

In this case, the equilibrium becomes limiting and the particle is just on the point of sliding down.

Hence we have the following theorem:

If a body be placed on a rough inclined plane and be on the point of sliding down the plane under the plane under the action of its weight and the reaction of the plane only, the angle of inclination of the plane of the horizon is equal to the angle of friction.

The inclination ($\lambda = tan^{-1}\mu$) of the inclined plane when the body just begins to slips is called the angle of repose. Hence the above theorem is stated as:

The angle of repose of a rough inclined plane is equal to the angle of friction.

Important note: It should be noted that the angle of repose of a rough inclined plane is equal to the angle of friction, only when there are no external force acting on the body.

Equilibrium of a body on a rough inclined plane under a force parallel to the plane:

Theorem1 . *A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted upon by a force, parallel to the plane and along the line of greatest slope; to find the limits between which the force must lie.*

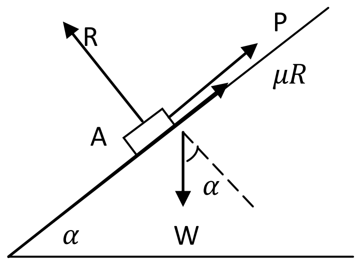


Fig.5

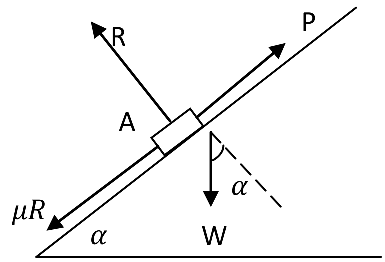


Fig.6

Let α be the inclination of the plane to the horizon, W the weight of the body and R the normal reaction.

Case 1. Refer fig. 5. Let the body be on the point of moving down the plane. Then limiting friction acts up the plane and $= \mu R$. Let P be the force required to keep the body at rest.

Resolving along and perpendicular to the plane, we have

$P + \mu R = W \sin \alpha$ …………………..(1)

And $R = W \cos \alpha$ ………………….(2)

Substituting for R from (2) in (1), we get

$$P = W \sin \alpha - \mu W \cos \alpha$$

If λ is the angle of friction, we know that $\mu = \tan \lambda$.

$$\therefore P = W(\sin \alpha - \tan \lambda \cos \alpha)$$

$$= W \frac{(\sin \alpha \cos \lambda - \sin \lambda \cos \alpha)}{\cos \lambda}$$

$= W \frac{\sin(\alpha - \lambda)}{\cos \lambda}$ Let this value of P be P_1,

$P_1 = W \frac{\sin(\alpha - \lambda)}{\cos \lambda}$ ………(3)

Since $\alpha > \lambda$, P_1 is positive.

Case2. As in fig. 6, let the body be on the point of moving up the plane. Then limiting friction μR acts downwards. Let P be the force required to keep the body at rest.

Resolving as before, we get,

$P - \mu R = W \sin\alpha$ …………….(4)

and $R = W\cos\alpha$ ………………(5)

Hence

$P = \mu W\cos\alpha + W\sin\alpha$

$= W(\tan\lambda\cos\alpha + \sin\alpha)$

$= W\frac{(\sin\lambda\cos\alpha - \sin\alpha\cos\lambda)}{\cos\lambda}$

$= W\frac{\sin(\alpha+\lambda)}{\cos\lambda}$

$W\frac{\sin(\alpha+\lambda)}{\cos\lambda} = P_2$ (Say)

$\therefore P_2 = W\frac{\sin(\alpha+\lambda)}{\cos\lambda}$ ……….(6)

Now if P is > P_2, the body will move up the plane.

$\therefore P_2$ is the limiting value of P, which is necessary to keep the body in equilibrium, without moving upwards.

If P is > P_1, the body will move down the plane.

$\therefore P_1$ is the limiting value of P, which is necessary to keep the body in equilibrium, without moving downwards.

Hence, if P lies between P_1 and P_2, the body will remain in equilibrium and is not in the point of motion in either direction.

Hence, for equilibrium, the force P must lie between the values $W\frac{\sin(\alpha-\lambda)}{\cos\lambda}$ and $W\frac{\sin(\alpha+\lambda)}{\cos\lambda}$.

Note. The value of P_2 may be obtained from that of P_1, by changing the sign of μ.

Equilibrium of a body on a rough inclined plane under any force.

Theorem2. *A body is at rest on a rough inclined plane of inclination α to the horizon, being acted on by a force making an angle θ with the plane; to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag to the body up the inclined plane.*

Notes

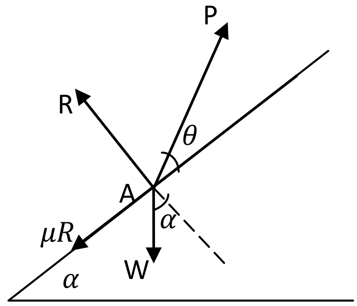


Fig. 7

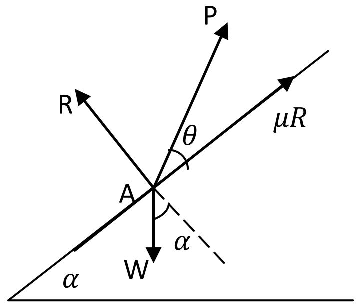


Fig. 8

Let W be the weight of the body, P the forec acting at an angle θ with the plane and R the normal reaction.

Caes I. In fig. 7, the body in just on the point of moving down the plane. Then limiting friction μR acts upwards. Resolving the forces along and perpendicular to the plane, we get
$P\cos\theta + \mu R = W\sin\alpha$ ………..(1)
And $P\sin\theta + R = W\cos\alpha$ ………..(2)
Substituting the value of R from (2) in (1), we get

$$P\cos\theta + \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$$

i.e.

$$P(\cos\theta - \mu\sin\theta) = W$$

$$\therefore P = W\frac{(\sin\alpha - \mu\cos\alpha)}{\cos\theta - \mu\sin\theta}$$

If λ is the angle of friction, we know that $\mu = \tan\lambda$.

$$\therefore P = W\frac{(\sin\alpha - \tan\lambda\cos\alpha)}{\cos\theta - \tan\lambda\sin\theta}$$

$$= W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

Let this value of P be P_1,
$\therefore P_1 = W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$ ……….(3)

Case2. As in fig. 8, the body just in the point of moving up the plane. Then limiting friction μR acts downwards, resolving the forces as before, Resolving as before,

$P\cos\theta - \mu R = W\sin\alpha$ …………….(4)
and $P\sin\theta = W\cos\alpha$ ………………(5)

Substituting the value of R from (5) in (4), we get
$P\cos\theta - \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$.
i.e. $P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$

$\therefore P = W \frac{(\sin\ \alpha+\mu \cos\ \lambda)}{\cos\ \theta+\mu \sin\ \theta}$

$$= W \frac{(\sin\alpha + \tan\lambda\cos\lambda)}{\cos\theta + \tan\lambda\sin\theta}$$

$= W \frac{\sin(\alpha+\lambda)}{\cos\ (\theta-\lambda)}$

Let this value of P be P_2,

$\therefore P_2 = W \frac{\sin(\alpha+\lambda)}{\cos\ (\theta-\lambda)}$ ……….(6)

P_1 and P_2, are the limiting values of the forces P, necessary to keep the body in equilibrium.

Hence if P lies between P_1 and P_2, the body will remain in equilibrium.

Corollary. We can find the direction and magnitude of the least force required to drag the body up the inclined plane.

From case II, $P = W \frac{\sin(\alpha+\lambda)}{\cos\ (\theta-\lambda)}$

Since α, W and λ are constants,

P is least, if $\cos(\theta - \lambda)$ is greatest.

i.e. if $\cos(\theta - \lambda) = 1$.

This happens when $\theta - \lambda = 0$. i.e. when $\theta = \lambda$.

In that case, value of $P = W\sin(\alpha + \lambda)$

Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

This result is something stated as:

"The best angle of traction up a rough inclined plane is the angle of friction."

8.10 PROBLEMS

Example 1. *A particle of weight 30kgs. Resting on a rough horizontal plane is just on the point of motion when acted on by horizontal forces of 6 kg.wt. and 8kg. wt. at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.*

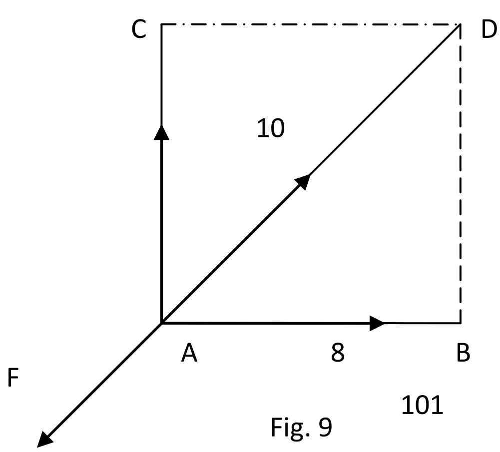


Fig. 9

Let AB (=8) and AC(=6) represent the directions of the forces, A begin the particle.

The resultant force $= \sqrt{8^2 + 6^2} = 10kgs.$ wt. and this acts along AD, making an angle $cos^{-1}\left(\frac{4}{5}\right)$.

The particle tends to move in the direction AD of the resultant force and hence friction acts in the opposite direction DA.

Let F be the frictional force. As motion just begins, magnitude of F is equal to that of that of the resultant force.

$\therefore F = 10$ ……(1)

If R is the normal reaction on the particle,

$R = 30$ ……(2)

If μ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$.

i.e. $10 = \mu.30$.

or $\mu = \frac{10}{30} = \frac{1}{3}$.

Check your Process

1. Define friction .
2. Explain Statical and dynamical friction.
3. Define law of dynamical friction.

8.11 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction; the force exerted is called the force is **friction**.
2. When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called **statical friction**. When motion ensures by one body sliding over another, the friction exerted is called **dynamical friction.**
3. When motion ensures by one body sliding over the other, the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

8.12 SUMMARY

- *If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction; the force exerted is called the force is* ***friction.***
- When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called **statical friction**.
- When one body is just on the point of sliding on another, the friction attains its maximum value and is called **limiting friction**.
- The equilibrium in this case is said to be **limiting.**

- When motion ensures by one body sliding over another, the friction exerted is called **dynamical friction.**
- *When motion ensures by one body sliding over the other, the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.*

8.13 KEYWORDS

- **Friction:** If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction; the force exerted is called the force is friction.
- **Statical friction***:* When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called statical friction.
- **Limiting friction**: When one body is just on the point of sliding on another, the friction attains its maximum value and is called limiting friction.
- **Limiting:** The equilibrium in this case is said to be limiting.
- **Dynamical friction**: When motion ensures by one body sliding over another, the friction exerted is called dynamical friction.

8.14 SELF ASSESSMENT QUESTIONS AND EXERICES

1. State the law of statical friction.
2. State the law of dynamical friction.
3. Comment on the statement that the “friction obstructs motion; friction helps motion”.
4. Comment on the statement that the “friction is passive resistance”.
5. A body of weight 4kgs, rests in limiting equilibrium on a rough plane whose slope is 30°. Find the coefficient of friction and the normal reaction.

8.15 FURTER READINGS

- Dr. M.K Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.
- Dr. M.K Venkataraman, Dyanamics, Agasthiar Publications, 13th Edition, 2009.
- P. Duraipandian, Laxmi Duraipandian & Muthamizh Jayapragasam, Mechanics,S.Chand&Co.Pvt.Ltd,2014.

BLOCK III
CATENARY, PROJECTILES AND IMPULSIVE FORCES

UNIT-IX CATENARY

STRUCTURE

9.0 Introduction

9.1 Objectives

9.2 Uniform String Under The Action Of Gravity

9.3 Equation Of The Common Catenary

9.4 Definitions

9.5 Tension at any point

9.6 Important formulae

9.7 Geometrical properties of the common catenary

9.8 Worked examples

9.9 Answers to Check Your Progress Questions

9.10 Summary

9.11 Keywords

9.12 Self Assessment Questions and Exercises

9.13 Further Readings

9.0 INTRODUCTION

In this chapter, we shall consider the equilibrium of perfectly flexible chains or strings of very small cross-section. A perfectly flexible string offers no resistance to beginning at any point. In such case, the action across any section of the string is a single force whose line of action is along the tangent to the curve formed by the string. A chain whose links are short and perfectly smooth, behaves like a flexible string.

9.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by catenary

- Discuss the action of gravity on a uniform string
- Discuss the geometrical properties of a catenary

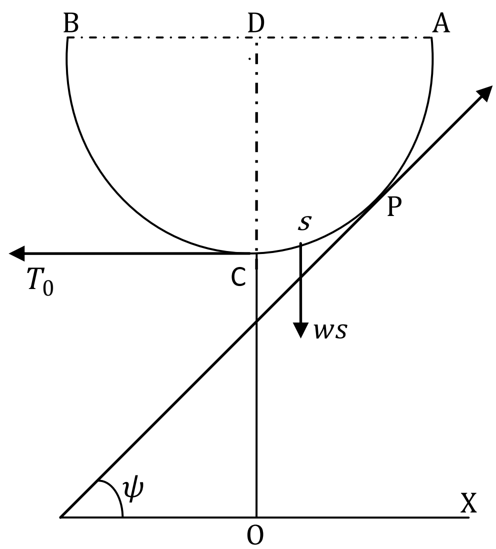
9.2 UNIFORM STRING UNDER THE ACTION OF GRAVITY

When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a *catenary* . If the weight per unit length of the chain or string is constant, the catenary is called the *uniform or common catenary*.

9.3 EQUATION OF THE COMMON CATENARY

A uniform heavy inextensible string hangs freely under the action of gravity; to find the equation of the curve which it forms.

Let ACB be a uniform heavy flexible cord attached to two points A and B at the same level, C being the lowest of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as y axis. Let P be any point of the string so that the length of the arc $CP = s$.



Consider the equilibrium of the portion CP of the chain.

The force acting on it are:

- Tension T_0 acting along the tangent at C and which is therefore horizontal.
- Tension T acting at Palong the tangent at P making an angle ψ with OX.
- Its weight ws acting vertically downwards through the C. G. of the arc CP.

For equilibrium, these three forces must be concurrent.

Hence the line of action of the weight ws must pass through the point of intersection of T and T_0.

Resolving horizontally and vertically, we have

$$T\cos\psi = T_0. \quad \ldots\ldots\ldots\ldots\ldots (1)$$

And $$T\sin\psi = ws \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Dividing (2) by (1), $\tan\psi = \frac{ws}{T_0}$.

Now it will be convenient to write the values of T_0 the tension at the lowest point, as $T_0 = wc \quad \ldots\ldots\ldots\ldots\ldots (3)$ where c is a constant. This means that we assume T_0, to be equal to the weight of an unknown length c of the cable.

Then $\tan\psi = \frac{ws}{wc} = \frac{s}{c}$.

$$\therefore s = c\tan\psi \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Equation (4) is called the *intrinsic equation* of the catenary.

It gives the relation between the length of the arc of the curve from the lowest point to any other point on the curve and the inclination of the tangent at the latter point.

To obtain the *Cartesian equation* of the catenary,

We use the equation (4) and the relations

$\frac{dy}{ds} = \sin\psi$ and $\frac{dy}{dx} = \tan\psi$ Which are true for any curve.

Now, $\frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi}$

$$= \sin\psi \frac{d}{d\psi} c\tan\psi$$

$$= \sin\psi \; c\sec^2\psi = c\sec\psi\tan\psi$$

$$\therefore y = \int c\sec\psi\tan\psi \, d\,\psi + A$$

$$= c\sec\psi + A$$

$y = c$ when $\psi = 0$, then $c = c\sec 0$.

$$\therefore A = 0$$

Hence $$y = c\sec\psi \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$\therefore y^2 = c^2\sec^2\psi = c^2(1+\tan^2\psi)$$

$$y^2 = c^2 + s^2 \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

$$\frac{dy}{dx} = \tan\psi = \frac{s}{c} = \frac{\sqrt{y^2 - c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c}$$

Integrating, $\cos h^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + B$.

When $x = 0, y = c$.

$$i.e. \cos h^{-1} 1 = 0 + B \; or \; B = 0$$

$\therefore \cos h^{-1}\left(\frac{y}{c}\right) = \frac{x}{c}$.

i.e. $y = c \cos h(\frac{x}{c})$ $\qquad \ldots\ldots\ldots\ldots\ldots (7)$

(7) is the Cartesian equation to the catenary.

We can also find the relation connecting s and x.

Differentiating (7).

$$\frac{dy}{dx} = c \sinh \frac{x}{c} . \frac{1}{c} = \sinh \frac{x}{c}$$

From (4), $s = c \tan\psi = c . \frac{dy}{dx} = c \sinh \frac{x}{c}$ $\qquad \ldots\ldots\ldots\ldots\ldots (8)$

9.4 DEFINITIONS

The Cartesian equation to the catenary is $y = c \cosh \frac{x}{c}$. $\cosh \frac{x}{c}$ is an even function of x. Hence the curve is symmetrical with respect to the y-axis i.e. to the vertical through the lowest point. This line of symmetry is called the axis of the catenary.

Since c is the only constant in the equation, it is called the *parameter* of the catenary and it determines the size of the curve.

The lowest point C is called the *verted* of the catenary. The horizontal line at a depth c below the vertex (which is taken by us the x-axis) is called the *directrix* of the catenary.

If the two points A and B from where the string is suspended are in a horizontal line, then the distance AB is called the *span* and the distance CD (i.e. the depth of the lowest point C below AB) is called the *sag*.

Notes

9.5 TENSION AT ANY POINT

We have derived the equations

$T\cos\psi = T_o$ …. … … (1)

and $T\sin\psi = ws$ …. … ….. (2)

we have also put $T_o = wc$ … … …. (3)

Equation (3) shows that the tension at the lowest point is a constant and is equal to the weight of a portion of the string whose length is equal to the parameter of the catenary. From equation (1), we find that the horizontal component of the tension at any point on the curve is equal to the tension at the lowest point and hence is a constant.

From equation (2), we deduce that the vertical component of the tension at any point is equal to ws i.e. equal to the weight of the portion of the string lying between the vertex and the point. ($\because s = arc\ CP$)

Squaring (1) and (2) and then adding,

$$T^2 = {T_o}^2 + w^2s^2$$

$$= w^2c^2 + w^2s^2$$

$$= w^2(c^2 + s^2)$$

$$= w^2y^2 \text{ using equation (6) of 9.3}$$

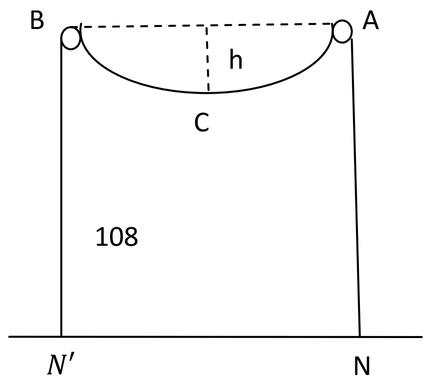
$$\therefore T = wy \qquad \text{… … …. (4)}$$

Thus the tension at any point is proportional to the height of the point above the origin. . It is equal to the weight of a portion of the string whose length is equal is equal to the height of the point above the directrix.

Important corollary:

Suppose a long chain is thrown over two smooth pegs A and B and is in equilibrium with the portions AN and BN' hanging vertically. The potion BCA of the chain will from a catenary.

The tension of the chain is unaltered by passing over the smooth peg A. The



tension at A can be calculated by two methods.

On one side (i.e. from the catenary portion) tension at A $= w.y$ where is the height of A above the directrix.

On the other side, tension at A=weight of the free part AN hanging down

and so it $= w.AN$

$$\therefore y = AN$$

In other words, N is on the directrix of the catenary.

Similarly N' is on the directrix.

Hence *if a long chain is thrown over two smooth pegs and is in equilibrium, the free ends must reach the directrix of the catenary formed by it*

9.6 IMPORTANT FORMULAE

The Cartesian coordinates of a point P on the catenary are (x, y) and its intrinsic coordinates are (s, ψ). Hence there are four variable quantities and we can have a relation connecting any two of them. There will be $4C_2 = 6$ such relations, most of them having been already derived. We shall derive the remaining. It is worthwhile to collect these results for ready reference.

(i) The relation connecting x and y is

$$y = c\cosh\frac{x}{c} \qquad \text{……… (1)}$$

and this is the Cartesian equation to the catenary.

(ii) The relation connecting s and ψ is $s = c\tan\psi$ ……… (2)

(iii) The relation connecting y and ψ is $y = c\sec\psi$ ……… (3)

(iv) The relation connecting y and s is $y^2 = c^2 + s^2$ …… (4)

(v) The relation connecting s and x is $s = c\sinh\frac{x}{c}$ …… (5)

The relations (1) and (5) have all been derived in 9.3.

(vi) We have to get the relation between x and ψ.

Since $y = c\cosh\frac{x}{c}$ and $y = c\sec\psi$,

we have $\sec\psi = \cosh\frac{x}{c}$

$$\therefore \frac{x}{c} = \cosh^{-1}(\sec\psi)$$

$$= \log\left(\sec\psi + \sqrt{\sec^2\psi - 1}\right)$$

$$= \log(\sec\psi + \tan\psi)$$

$$\therefore x = c\log(\sec\psi + \tan\psi) \quad \text{…..} (6)$$

This relation can also be obtained thus:

$$\frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi}$$

$= cos\,\psi\,.\frac{d}{d\psi}(c\,tan\,\psi)$ since $\frac{dx}{ds} = cos\,\psi$ for any curve $= cos\,\psi\;.c\,sec^2\psi = c\sec\psi$

Integrating, $x = \int c\sec\psi\; d\psi + D$

$$= c\log(\sec\psi + \tan\psi) + D$$

At the lowest point, $\psi = 0$ and $x = 0$

$\therefore 0 = c\log(\sec 0 + \tan 0) + D$

i.e. $0 = D$

$\therefore \quad x = c\log(\sec\psi + \tan\psi)$

(vii) The tension at any point $= wy$ …. … … .. (7)

where y is the distance of the point from the directrix.

(viii) The tension at the lowest point $= wc$ … … … (8)

By using the formulae (1) to (8), given in this section, we can solve most problems on the catenary.

We recall the following logarithmic expansions of the inverse hyperbolic functions, which will be frequently used in this chapter.

$$\sinh^{-1}x = \log(x + \sqrt{x^2+1})$$

$$\cosh^{-1}x = \log(x + \sqrt{x^2-1})$$

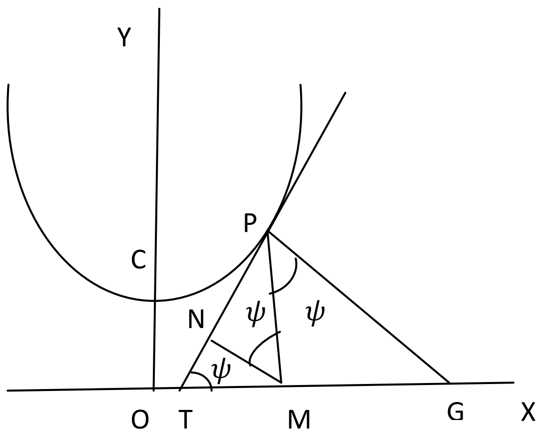
9.7 GEOMETRICAL PROPERTIES OF THE COMMON CATENARY

Let P be any point on the catenary $y = c\cosh\frac{x}{c}$.

PT is the tangent meeting the directrix (i.e. the x-axis) at T.

$$\angle PTX = \psi$$

PM (=y) is the ordinate of P and PG is the normal at P.



Draw $MN \perp to\ PT.$

From $\triangle PMN, MN = PM \cos\psi$

$$= y \cos\psi$$

$$= c \sec\psi \cos\psi$$

$$= c = constant$$

i.e. The length of the perpendicular from the foot of the ordinate on the ordinate on the tangent at any point of the catenary is constant.

Again $\tan\psi = \frac{PN}{MN} = \frac{PN}{c}$

$$\therefore \quad PN = c \tan\psi = s = arc\ CP$$

$$PM^2 = NM^2 + PN^2$$

$\therefore\ y^2 = c^2 + s^2,$ a relation already obtained.

If ρ is the radius of curvature of the catenary at P,

$$\rho = \frac{ds}{d\psi} = \frac{d}{d\psi}(c \tan\psi) = c \sec^2\psi$$

Let the normal at P cut the x-axis at G.

Then $PG. \cos\psi = PM = y$

$$\therefore \quad PG = \frac{y}{\cos\psi} = c \sec\psi . \sec\psi = c \sec^2\psi$$

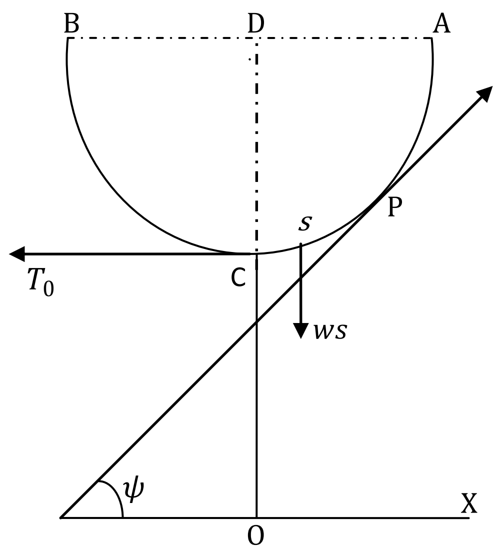
$$\therefore \rho = PG$$

Hence the radius of curvature at any point on the catenary is numerically equal to the length of the normal intercepted between the curve and the directrix, but they are drawn in opposite directions.

9.8 WORKED EXAMPLES

Example 1. A uniform chain of length l is to be suspented from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be $\frac{1}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$

Refer to figure. Let y_A and y_C be the y-coordinates of the highest point A and the lowest point C. Let w be the weight per unit length of the chain and c the parameter of the catenary.



Tension at A$= wy_A$

and tension at C $= w.y_C$ since

$T = wy$ at any point

Now $wy_A = n.w.y_C$

$$\therefore y_A = n.y_C = nc$$

But $y_A = c\cosh\frac{x_A}{c} = nc$

$$\therefore \cosh\frac{x_A}{c} = n$$

or $\frac{x_A}{c} = c \cosh^{-1} n = \log(n + \sqrt{n^2-1})$

$\therefore x_A = \text{c} \log(n + \sqrt{n^2-1})$ … …. … (1)

We have to find c.

${y_A}^2 = c^2 + {s_A}^2$, s_A denoting the length of CA.

$= c^2 + \frac{l^2}{4}$ (as total length =l)

i.e. $n^2c^2 = c^2 + \frac{l^2}{4}$

or $c^2 = \frac{l^2}{4(n^2-1)}$.

$\therefore c = \frac{l}{2\sqrt{n^2-1}}$ …. …. … (2)

Substituting (2) in (1),

$$x_A = \frac{l}{2\sqrt{n^2-1}} \log(n + \sqrt{n^2-1})$$

$\therefore$ span $AB = 2\ x_A = \frac{l}{\sqrt{n^2-1}} \log(n + \sqrt{n^2-1})$

Example 2.

A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite, the inclination of the wire to the ground is $2\tan^{-1}\frac{h}{l}$ and that its tensions there and at the ground are $\frac{w(l^2+h^2)}{2h}$ and $\frac{w(l^2-h^2)}{2h}$ where w is the weight of the wire per unit of length .

C is the vertex of the catenary CA, A being the kite. The origin O is taken at

a depth c below C.

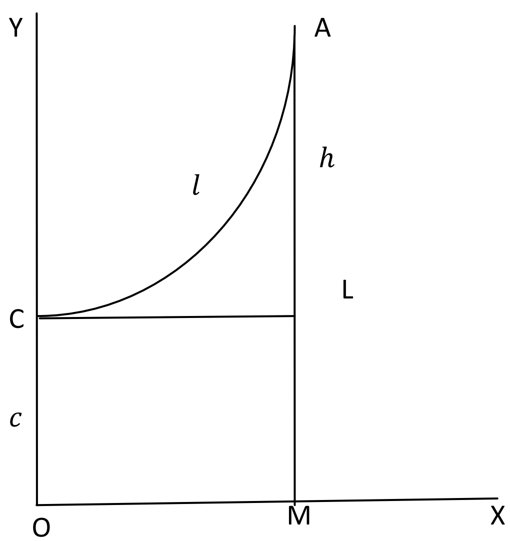
Then $y_A = c + h$ and $s_A = arc\ CA = l$

Since $y^2 = c^2 + s^2$, we have

$$(c+h)^2 = c^2 + l^2$$

i.e. $\quad h^2 + 2ch = l^2$

or $c = \frac{l^2-h^2}{2h}$ … …. … (1)



We know that $s = c\tan\psi$ … …. …. …. (2)

Applying (2) at the point A, we have

$l = c.\tan\psi_A$

$\therefore \tan\psi_A = -\frac{2hl}{l^2-h^2}$ substituting for c from (1)

$= \frac{2\left(\frac{h}{l}\right)}{1-\left(\frac{h}{l}\right)^2}$ ….. …… ….(3)

But $\tan\psi = \frac{2\tan\frac{\psi}{2}}{1-\tan^2\frac{\psi}{2}}$ ….. … ….. (4)

Comparing (3) and (4), we find that

$$\tan\frac{\psi}{2}\ at\ A = \frac{h}{l}$$

$$\therefore \frac{\psi}{2} = \tan^{-1}\frac{h}{l}$$

Or$\psi\ \ at\ A = 2\ \tan^{-1}\frac{h}{l}$

The tension at A$= w.y_A$

$$= w.(c+h)$$

$$= w\left(\frac{l^2-h^2}{2h} + h\right)$$

$$= w\frac{(l^2+h^2)}{2h}$$

Check Your Progress

1. Define catenary.
2. Define common catenary.
3. Write down the equation of the common catenary.
4. Write down the geometrical properties of the common catenary
5. Define directrix.

9.9 Answers to Check Your Progress Questions

1. When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a ***catenary*** .
2. If the weight per unit length of the chain or string is constant, the catenary is called the ***uniform or common catenary***.
3. The equation of the common catenary is $\boldsymbol{y = c\cosh(\frac{x}{c})}$.
4. The geometrical properties of the common catenary are

➢ *The length of the perpendicular from the foot of the ordinate on the ordinate on the tangent at any point of the catenary is constant.*

- *The radius of curvature at any point on the catenary is numerically equal to the length of the normal intercepted between the curve and the directrix, but they are drawn in opposite directions.*

5. The horizontal line at a depth c below the vertex (which is taken by us the x-axis) is called the **directrix** of the catenary.

9.10 SUMMARY

- When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a ***catenary*** . If the weight per unit length of the chain or string is constant, the catenary is called the ***uniform or common catenary***.
- The Cartesian equation to the catenary is $y = c\cosh\frac{x}{c}$. $\cosh\frac{x}{c}$ is an even function of x. Hence the curve is symmetrical with respect to the y-axis i.e. to the vertical through the lowest point. This line of symmetry is called the ***axis*** of the catenary.
- Since c is the only constant in the equation, it is called the ***parameter*** of the catenary and it determines the size of the curve.
- The lowest point C is called the ***verted*** of the catenary. The horizontal line at a depth c below the vertex (which is taken by us the x-axis) is called the ***directrix*** of the catenary.
- If the two points A and B from where the string is suspended are in a horizontal line, then the distance AB is called the ***span*** and the distance CD (i.e. the depth of the lowest point C below AB) is called the ***sag***.

9.11 KEYWORDS

Catenary: When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a ***catenary*** .

9.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. If T be the tension at any point P of the string and T_0, that the lowest point C, prove that $T^2 - T_0^2 = W$, W being the weight of the arc CP of the string.
2. Prove that, if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their weights.
3. A uniform chain of length l is to has its extremities fixed at two points in the same horizontal line. Shoe that the span must be $\frac{1}{\sqrt{8}}\log(3+\sqrt{8})$ in order that the tension at each support shall be tree times that at the lowest point.
4. A uniform chain of length l is suspended from two points A, B in the same horizontal line. If the tension at A is twice that at the lowest point, shw that the span AB is $\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$.
5. A chain of length $2l$ hangs between two points A and B on the same level. The tension both at A and B is 5 times the tension at the lowest point. Prove that the horizontal distance between A and B is $\frac{1}{\sqrt{6}}\log(5+2\sqrt{3})$.

Notes

9.13 FURTHER READINGS

1. Dr. M. K. Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.
2. Dr. M. K. Venkataraman, Dynamics, Agasthiar Publications, 13th Edition, 2009.
3. P. Duraipandian, Laxmi Duraipandian & Muthamizh Jayapragasam, Mechanics,S.Chand&Co.Pvt.Ltd,2014.

UNIT X PROJECTILE

STRUCTURE

10.0 Introduction
10.1 Objectives
10.2 Definition
10.3 Fundamental Principles
10.4 Path of the Projectile
10.5 Characteristics of the motion of a projectile
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10.0 INTRODUCTION

In this chapter we shall consider, motion of a particle projected into the air in any direction and with any velocity. Such a particle is called a projectile. The two forces that act on the projectile are its weight and the resistance of air. For simplicity, we suppose the motion to take place within such a moderate distance from the surface of the earth that we can neglect the variations in the acceleration due to gravity. This means that g may be considered to be constant in magnitude throughout the motion of the projectile. Secondly, we shall neglect the resistance of the air and consider the motion to take place in vacuum.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by projectile
- Discuss the properties of a projectile
- Discuss the properties of inclined plane.

10.2 DEFINITIONS

The following terms are used in connection with projectiles:

The *angle of projection* is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.

The *velocity of projection* is the velocity with which the particle is projected.

The *trajectory* is the path which the particle describes.

The *range* on a plane through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.

The *time of flight* is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

10.3 FUNDAMENTAL PRINCIPLES

To discuss the motion of a projectile, we consider the horizontal and vertical components of the motion separately. The only force acting on the projectile is gravity and this acts vertically downwards. Hence by the Physical Independence of forces, it has no effect on the horizontal motion of the particle. So the *horizontal velocity remains constant throughout the motion*, as there is no force to cause any acceleration in that direction. On the other hand, the weight of the particle acting vertically downwards, will have its full effect on the vertical motion of the particle. The weight mg acting vertically downwards on a particle of mass m will produce an acceleration g vertically downwards. Hence the vertical component of the velocity will be subject to a retardation g. These two main principles will help us to study the motion of a particle.

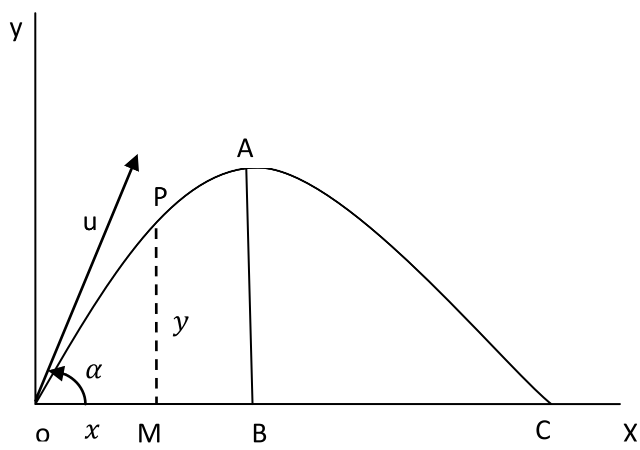
10.4 PATH OF THE PROJECTILE

Let a particle from O, with a velocity u at an angle α to the horizon. Take O as the origin, the horizontal and the upward vertical through O as axes of x and y respectively. The initial velocity u can be split into two components, which are $u\cos\alpha$ in the horizontal direction and $u\,sin\,\alpha$ in the vertical direction. The horizontal component $u\cos\alpha$ is constant throughout the motion as there is no horizontal acceleration The vertical component $u\sin\alpha$ is subject to an acceleration g downwards.

Let $P(x, y)$ be the position of the particle at time t secs. after projection. Then

$x =$ horizontal distance described in t secs. $= (u\ \cos\alpha).t$ (1)

$y =$ vertical distance described in t secs. $= (u\ \sin\alpha).t$ (2)



(1) and (2) can be taken as the parametric equations of the trajectory. The equation to the path is got by eliminating t between them.

From (1), $t = \frac{x}{u \cos \alpha}$ and putting this in (2) we get

$$y = u \sin \alpha \,.\frac{x}{u \cos \alpha} - \frac{1}{2} g \,.\left(\frac{x}{u \cos \alpha}\right)^2$$

i.e. $$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha} \qquad (3)$$

Multiplying (3) by $2u^2 \cos^2 \alpha$,

$2u^2 \cos^2 \alpha \,.\, y = 2u^2 \cos^2 \alpha \,.\, x \frac{\sin \alpha}{\cos \alpha} - g x^2$

i.e. $x^2 - \frac{2u^2 \sin \alpha \cos \alpha}{g} x = -\frac{2u^2 \cos^2 \alpha}{g} y$

(or) $\left(x - \frac{u^2 \sin \alpha \cos \alpha}{g}\right)^2 = \frac{u^4 \sin^2 \alpha \cos^2 \alpha}{g^2} - \frac{2u^2 \cos^2 \alpha}{g} y$

$$= -\frac{2u^2 \cos^2 \alpha}{g} \left(y - \frac{u^2 \sin^2 \alpha}{2g}\right)$$

Transfer the origin to the point

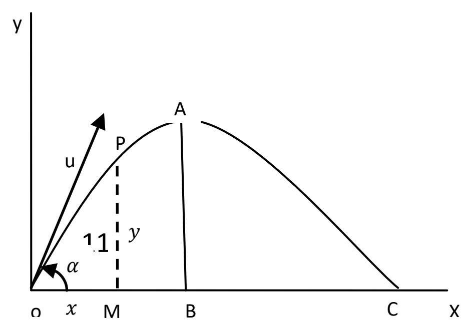
$\left(\frac{u^2 \sin \alpha \cos \alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g}\right)$.

The above equation then becomes

$$X^2 = -\frac{2u^2 \cos^2 \alpha}{g} . Y$$

(4) is clearly the equation to a parabola of latus rectum $\frac{2u^2 \cos^2 \alpha}{g}$, whose axis is vertical and downwards and whose vertex is the point $\left(\frac{u^2 \sin \alpha \cos \alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g}\right)$.

Note. The latus rectum of the above parabola is



$$= \frac{2u^2 \cos^2 \alpha}{g} = \frac{2}{g}.(u\ \cos\alpha)^2$$

$$= \frac{2}{g} \times square\ of\ the\ horizontal\ velocity$$

So the latus rectum (i.e. the size of the parabola) is independent

of the initial vertical velocity and depends only on the horizontal velocity.

10.5 CHARACTERISTICS OF THE MOTION OF A PROJECTILE

Let a particle be projected from $O$with velocity u at an angle α to the horizontal OX. Let A be the highest point of the path and C the point where it again meets the horizontal plane through$O$. Using the two fundamental principles, we can derive the following results relating to the motion of a projectile.

(1) Greatest height attained by a projectile.

At A, the highest point, the particle will be moving only horizontally, having lost all its vertical velocity. Let $AB = h$ =the greatest height reached. Considering vertical motion separately, initial upward vertical velocity $= u\ \sin\alpha$ and the acceleration in this direction is $-g$. The final vertical velocity at A is $= 0$.

Hence $0 = (u\ \sin\ \alpha)^2 - 2g.h$ i.e. $h = \frac{u^2 \sin^2 \alpha}{2g}$

i.e. the vertex of the parabola is the highest point of the path.

(2) *Time taken to reach the greatest height.*

Let T be the time from O to A. Then, in time T, the initial vertical velocity $u\ \sin\alpha$ is reduced to zero, acted on by an acceleration $-g$. Hence $0 = u\ \sin\alpha - \mathrm{gT}$.

$\therefore\ \ T = \frac{u\ \sin\ \alpha}{g}$.

(3)*Time of flight i.e. the time taken to return to the same horizontal level as O.*

When the particle arrives at O, the effective vertical distance it has described is zero. Hence if t is the time of flight, considering vertical motion, we have $0 = u\ \sin\alpha\,.\,t - \frac{1}{2}gt^2$.

i.e. $t = 0$ (or) $t = \frac{2u\ \sin\ \alpha}{g}$

$t = 0$ is the instant of projection when also the vertical distance travelled is zero.

$$\therefore \quad The\ time\ of\ flight = \frac{2u\ \sin\alpha}{g}$$

We find the time of flight is twice the time taken to reach the highest point, as we should expect from symmetry.

(4)*The range on the horizontal plane through the point of projection .*

The time of flight is $t = \frac{2u\ \sin\alpha}{g}$. During this time, the horizontal velocity remains constant and is equal to $u\ \cos\alpha$.

Hence $OC =$ horizontal distance described in time t

$$= u\ \cos\alpha . t = u\ \cos\alpha . \frac{2u\ \sin\alpha}{g} = \frac{2u^2\ \sin\alpha\cos\alpha}{g}$$

Hence the horizontal range $R = \frac{2u^2\ \sin\alpha\cos\alpha}{g} = \frac{u^2\ \sin 2\alpha}{g}$

Note. (1) The horizontal range can also be found thus: The equation to the path is $\quad y = x\tan\alpha - \frac{gx^2}{2u^2\ \cos^2\alpha}$ .... (1)

The equation to the x axis is $y = 0$.

Putting $y = 0$ in (1), we have $x\tan\alpha - \frac{gx^2}{2u^2\ \cos^2\alpha} = 0$

i.e. $x = 0$ (or) $x = \frac{2u^2\ \cos^2\alpha\tan\alpha}{g} = \frac{2u^2\ \sin\alpha\cos\alpha}{g}$

$x = 0$ corresponds to the point of projection and so the other value $\frac{2u^2\ \sin\alpha\cos\alpha}{g}$ gives the horizontal range.

(1) horizontal range= $\frac{2u^2\ \sin\alpha\cos\alpha}{g} = \frac{2(u\ \cos\alpha).(u\ \sin\alpha)}{g}$

$= 2\frac{UV}{g}$ where U and V are the initial horizontal and vertical velocities.

Example 1. A body is projected with a velocity of 98 metres per sec. in a direction making an angle $\tan^{-1} 3$ with the horizon; show that it rises to a vertical height of 441 metres and that its time of flight is about 19 secs. Find also horizontal range through the point of projection(g=9.8 metres/sec^2).

Here $u = 98$; n $\alpha = \tan^{-1} 3$ i.e. $\tan\alpha = 3$.

$$\therefore \quad \sin\alpha = \frac{\sin\alpha}{\cos\alpha}.\cos\alpha = \frac{\tan\alpha}{\sec\alpha} = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}} = \frac{3}{\sqrt{10}}$$

$$\cos\alpha = \frac{\sin\alpha}{\tan\alpha} = \frac{1}{\sqrt{10}}$$

Greatest height reached $= \frac{u^2 \sin^2\alpha}{2g} = \frac{98\times98\times9}{10\times2\times9.8}$

$$= 441\ metres.$$

Time of flight $= \frac{2u\sin\alpha}{g} = \frac{2\times98\times3}{\sqrt{10}\times9.8} = 6\sqrt{10}$

$= 6\times3.162 = 18.972 = 19\ secs.$ nearly

Horizontal range $= \frac{2u^2\sin\alpha\cos\alpha}{g}$

$$= \frac{2\times98\times98}{9.8}\times\frac{3}{\sqrt{10}}\times\frac{1}{\sqrt{10}} = 588\ metres.$$

Example 2. If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection, find the angle of projection.

Let u be the initial velocity and α the angle of projection.

Then, the greatest height $= \frac{u^2\sin^2\alpha}{2g}$

and horizontal range $= \frac{2u^2\sin\alpha\cos\alpha}{g}$

It is given that $\frac{u^2\sin^2\alpha}{2g} = \frac{1}{4}\times\frac{2u^2\sin\alpha\cos\alpha}{g}$

i.e. $\frac{u^2\sin^2\alpha}{2g} = \frac{u^2\sin\alpha\cos\alpha}{2g}$

i.e. $\sin\alpha = \cos\alpha$ (or) $\tan\alpha = 1$ $\therefore\ \alpha = 45^\circ$

Example 3. Show that the greatest height which a particle with initial velocity v can reach on a vertical wall at a distance 'a' from the point of projection is $\frac{v^2}{2g} - \frac{g\,a^2}{2v^2}$. Prove also that the greatest height above the point of projection attained by the particle in its flight is $v^6/2g(v^4 + g^2a^2)$.

In the usual notation, the equation to the path is

$$y = x\tan\alpha - \frac{g\,x^2}{2v^2\cos^2\alpha} \qquad \ldots\ldots\ (1)$$

Putting $x = a$ in (1), we get the value of y, which is the height reached on the vertical wall at a distance 'a' from the point of projection.

$$\therefore y = a\tan\alpha - \frac{g\,a^2}{2v^2\cos^2\alpha}$$

$= at - \frac{g\,a^2}{2v^2}(1+\ t^2)$ where $t =\ \tan\alpha$……(2)

Now a and v are given and so y is a function of t.

$\therefore y$ is maximum when $\frac{dy}{dt} = 0$ and $\frac{d^2y}{d^2t}$ is negative.

Differentiating (2) with respect to t,

$$\frac{dy}{dt} = a - \frac{g\,a^2}{2v^2}.2t = a - \frac{g\,a^2t}{v^2}$$

$\frac{d^2y}{d^2t} = -\frac{g\,a^2}{v^2} =$ negative clearly.

So y is maximum when $a - \frac{g\,a^2t}{v^2} = 0$ (or) $t = \frac{v^2}{ga}$ …….(3)

Putting this value of t in (2),

max. value of $y = a.\frac{v^2}{ga} - \frac{g\,a^2}{v^2}(1+\frac{v^4}{g^2\,a^2})$

$$= \frac{v^2}{g} - \frac{g\,a^2}{2v^2} - \frac{v^2}{2g} = \frac{v^2}{2g} - \frac{g\,a^2}{2v^2}.$$

This is the greatest height reached on the wall.

Greatest height attained during the flight

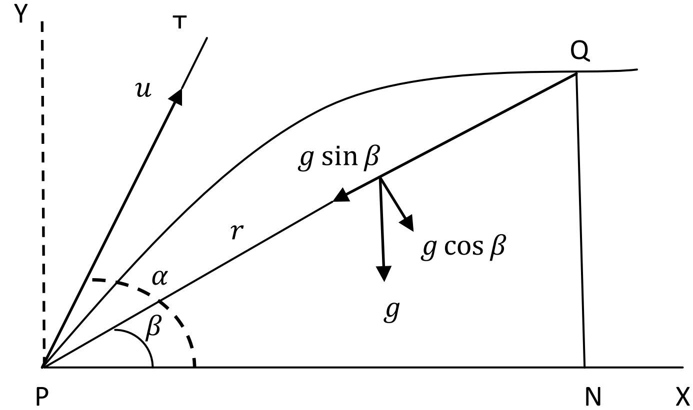
$$= \frac{v^2\sin^2\alpha}{2g} = \frac{v^2}{2g}.\frac{1}{\operatorname{cosec}^2\alpha} = \frac{v^2}{2g(1+\cot^2\alpha)}$$

$= \frac{v^2}{2g\ (1+\frac{g^2\,a^2}{v^4})}$ putting the value of $\tan\alpha$ from (3)

$$= \frac{v^6}{2g(v^4+g^2\,a^2)}$$

10.6 RANGE ON AN INCLINED PLANE

From a point on a plane, which is inclined at an angle β to the horizon, a particle is projected with a velocity u at an angle α with the horizontal, in a plane passing through the normal to the inclined plane and the



line of greatest slope. To find the range on the inclined plane.Let P be the point of projection and the particle strike the inclined plane at O. Then PQ is the range on the inclined plane. Let $PQ = r$. Taking P as the origin and horizontal and the vertical through P as the axe of $x$and v respectively the equation to the path is,

$$y = x \tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \ldots\ldots\ldots. (1)$$

Draw $QN \perp$ to the horizontal plane through P. The co-ordinates of Q are $(r\cos\beta, r\sin\beta)$. Substituting these in (1),

$$r\sin\beta = r\cos\beta\,.\tan\alpha - \frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$$

Multiplying by $2u^2\cos^2\alpha$ and canceling r throughout, we have

$$2u^2\cos^2\alpha\sin\beta = 2u^2\cos\beta\ \sin\alpha\cos\alpha - gr\ \cos^2\beta$$

$$\therefore r = \frac{2u^2\cos\beta\ \sin\alpha\cos\alpha - 2u^2\ \cos^2\alpha\ \sin\beta}{g\ \cos^2\beta}$$

$$= \frac{2u^2\cos\alpha(\sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta)}{g\ \cos^2\beta}$$

i.e. $r = \dfrac{2u^2\cos\alpha\ \sin(\alpha-\beta)}{g\ \cos^2\beta}$

Aliter: We can study separately the motion of the particle along the inclined plane and the motion perpendicular to the plane. The initial velocity u can be resolved into two components (i) $u\ \cos(\alpha-\beta)$ along PQ, the inclined plane and (ii) $u\ \sin(\alpha-\beta)$, perpendicular to the inclined plane. The acceleration g can be resolved into two components (i) $g\cos\beta$ perpendicular to the inclined plane in the downwards direction and (ii) $g\sin\beta$ along the inclined plane towards P. This resolution is shown in the figure . Let T be the time which the particle takes to go from P to Q. After time T, the particle is again on the inclined plane and so, during time T, the distance travelled perpendicular to the inclined plane is $= 0$.

$$\therefore\ 0 = u\ \sin(\alpha-\beta)\ .\mathrm{T} - \frac{1}{2}\,\mathrm{g}\cos\beta\ .T^2$$

i.e. $T = \frac{2u\ \sin(\alpha-\beta)}{g\cos\beta}$

This is the time of flight on the inclined plane. During this time, the horizontal velocity remains constant and $= u\cos\alpha$. So horizontal distance described in time $T = PN = u\cos\alpha\ T$. But $PN = PQ\cos\beta$.

$$\therefore PQ\ .\cos\beta = u\cos\alpha\ T$$

i.e. $PQ = \frac{u\cos\alpha}{\cos\beta}.T\ =\ \frac{u\cos\alpha}{\cos\beta}\ \frac{2u\ \sin(\alpha-\beta)}{g\cos\beta}$

$= \frac{2\,u^2\ \sin(\alpha-\beta)\ \cos\alpha}{b\ \cos^2\beta}$

10.7 GREATEST DISTANCE MAXIMUM RANGE

To find the greatest distance of the projectile from the inclined plane and show that is attained in half the total time of flight:

Let us consider the motion perpendicular to the inclined plane. As explained in the above section, the initial velocity in this direction is $u\sin(\alpha-\beta)$ and this is subject to an acceleration $g\cos\beta$ in the same direction but acting downwards. Let y be the distance travelled by the particle in this direction in time t. Then

$y = u\sin(\alpha-\beta)\,.\,t - \frac{1}{2}\ g\cos\beta\,.\,t^2$ …………..(1)

Differentiating with respect to t,

$\frac{dy}{dt} =\ u\sin(\alpha-\beta) - g\cos\beta\,.\,t$ ………… (2)

and $\frac{d^2y}{d^2t} =\ -g\cos\beta$ =negative.

So y is maximum when $\frac{dy}{dt} = 0$

i.e. when $u\sin(\alpha-\beta) - g\cos\beta\ .\,t = 0$

i.e. $t = \frac{u\sin(\alpha-\beta)}{g\cos\beta}$ ………. (3)

Substituting (3) in (1), maximum value of y

Notes

$$= u\sin(\alpha-\beta)\ .\frac{u\sin(\alpha-\beta)}{g\cos\beta} - \frac{1}{2}\ g\cos\beta\,.\frac{u^2\sin^2(\alpha-\beta)}{g^2\cos^2\beta}$$

$$= \frac{u^2\sin^2(\alpha-\beta)}{g\cos\beta} - \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta} = \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta} \qquad \ldots\ldots\ldots\ldots(4)$$

(1) is the greatest distance of the projectile from the inclined plane.

Also, from (3), time to this greatest distance $= \frac{u\sin(\alpha-\beta)}{g\cos\beta}$ and this is clearly half of the time of flight.

Aliter: When the particle is at the greatest distance from the inclined plane, it will have all its velocity only parallel to the inclined plane . Hence the component velocity perpendicular to the inclined plane is zero. So, if s is the greatest distance, we have

$$0 = [u\sin(\alpha-\beta)]^2 - 2g\cos\beta\ .s$$

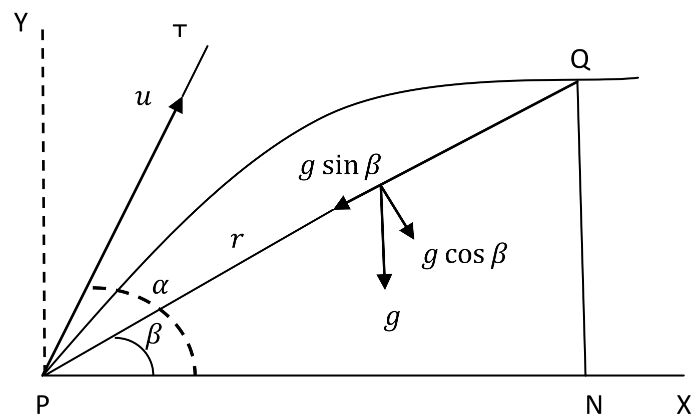
i.e. $s = \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta}$

Also, if t is the corresponding time,

$0 = u\sin(\alpha-\beta) - g\cos\beta\ t$ (or) $t = \frac{u\sin(\alpha-\beta)}{g\cos\beta}$

To determine when the range on the inclined plane is maximum, given the magnitude u of the velocity of projection.

The range R on an inclined plane is given by



$$R = \frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta} = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta] \qquad \ldots\ldots\ (1)$$

Now, u and β are given. The quantity outside the bracket, $\frac{u^2}{g\cos^2\beta}$ is constant. So R is maximum, when the value of the expression inside the bracket is a maximum.

i.e. when $\sin(2\alpha-\beta)$ is greatest.

i.e. when $2\alpha-\beta=\frac{\pi}{2}$.

i.e. $\alpha=\frac{\pi}{4}+\frac{\beta}{2}$ for maximum range.

When α takes this value,

$\alpha-\beta=(2\alpha-\beta)-\alpha=90^\circ-\alpha$ …….. (2)

Referring to figure,

$\alpha-\beta=\angle TPN-\angle QPN=\angle TPQ$ and $90^\circ-\alpha=\angle YPT$

Hence from (2), $\angle TPQ=\angle YPT$.

i.e. PT, *the direction of projection for maximum range bisects the angle between the vertical and the inclined plane.*

From (1), the value of maximum range

$$=\frac{u^2}{g\cos^2\beta}(1-\sin\beta)=\frac{u^2}{g\,(1+sin\,\beta)}$$

To show that, for a given initial velocity of projection, there are, in general, two possible directions of projection so as to obtain a given range on an inclined plane:

Let u be the velocity of projection of a particle and α the necessary angle of projection so as to get a given k on an inclined plane of inclination β to the horizontal.

Then

$k=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}=\frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta$ ………(1)

From (1), $\sin(2\alpha-\beta)=\frac{gk\cos^2\beta}{u^2}+\sin\beta$ ………. (2)

Since k,u,β are given, the R.H.S. of (2) is a known positive quantity. So we can determine an acute angle θ whose sine is exactly to $\frac{gk\cos^2\beta}{u^2}+\sin\beta$

Then (2) becomes, $\sin(2\alpha-\beta)=\sin\theta$ …….. (3)

i.e. $\qquad 2\alpha - \beta = \theta \quad \text{(or)} \quad \alpha = \frac{\theta}{2} + \frac{\beta}{2}$ $\qquad$ …….. (4)

Since $\sin(180^\circ - \theta) = \sin\theta$, (3) can also be written as $\sin(2\alpha - \beta) = \sin(180^\circ - \theta)$. Then $2\alpha - \beta = 180^\circ - \theta$

i.e. $\alpha = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2}$ $\qquad$ …….. (5)

From (4) and (5), we find that there are two value of α and so two directions of projection, each giving the same range k.

Let α_1 and α_2 these two values of α.

Then $\alpha_1 = \frac{\theta}{2} + \frac{\beta}{2}$ and $\alpha_2 = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2}$

Now $\left(45^\circ + \frac{\beta}{2}\right) - \alpha_1 = 45^\circ + \frac{\beta}{2} - \frac{\theta}{2} - \frac{\beta}{2} = 45^\circ - \frac{\theta}{2}$

and $\alpha_2 - \left(45^\circ + \frac{\beta}{2}\right) = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2} - 45^\circ - \frac{\beta}{2} = 45^\circ - \frac{\theta}{2}$

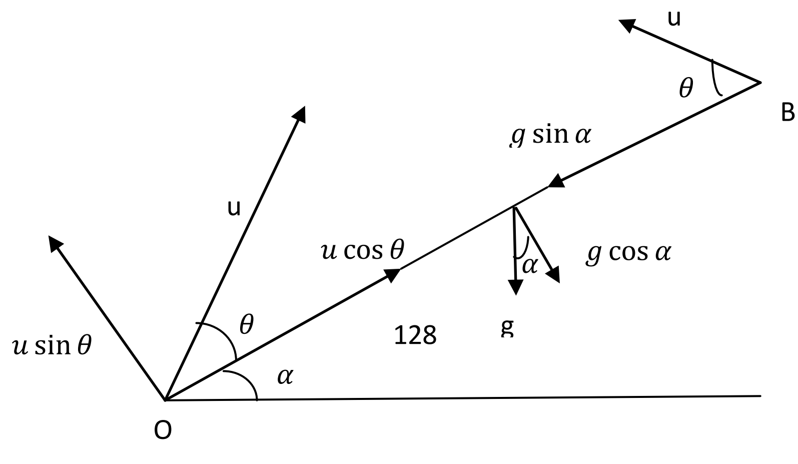
$\therefore \quad \left(45^\circ + \frac{\beta}{2}\right) - \alpha_1 = \alpha_2 - \left(45^\circ + \frac{\beta}{2}\right)$ $\qquad$ ……… (6)

But $45^\circ + \frac{\beta}{2}$ is the angle of projection for maximum range on the inclined plane. So, (6) shows that the two directions α_1 and α_2 are equally inclined to the direction of maximum range.

Example 1. Show that, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\frac{1+\sin\alpha}{1-\sin\alpha}$.

Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane. The velocity u can be resolved into two components $u\cos\theta$ along the upward inclined plane and $u\sin\theta$ perpendicular to the inclined plane. The acceleration g can be resolved into two components, $g\cos\alpha$ perpendicular to the inclined plane and downwards.

Consider the motion perpendicular to the inclined plane. Let T be the time flight.



Distance travelled perpendicular to the inclined plane in time T is $= 0$.

$$\therefore \quad 0 = u\sin\theta\,.T - \frac{1}{2}\,g\cos\alpha\,.T^2$$

$$\text{i.e. } T = \frac{2u\sin\theta}{g\cos\alpha}.$$

During this time, the distance travelled along the plane

$$= u\cos\theta\,.T - \frac{1}{2}\,g\sin\alpha\,.T^2$$

$$= u\cos\theta\,.\frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{2}\,g\sin\alpha\,.\frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$$

$$= \frac{2u^2\cos\theta\sin\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta - \sin\alpha\ \sin\theta).$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta+\alpha) = \frac{u^2}{g\cos^2\alpha}.2\cos(\theta+\alpha)\sin\theta$$

$$= \frac{u^2}{g\cos^2\alpha}[\sin(2\theta+\alpha) - \sin\alpha]$$

This is the range R_1 up the inclined plane.

R_1 is maximum, when $\sin(2\theta+\alpha) = 1$

$\therefore$ Maximum range up the plane

$$= \frac{u^2}{g\cos^2\alpha}[\,1 - \sin\alpha] = \frac{u^2}{g(1+\sin\alpha)}$$

when the particle is projected down the plane from B at the same angle to the plane, the time of flight has the same value $\frac{2u\sin\theta}{g\cos\alpha}$. But the component of the initial velocity along the inclined plane is $u\cos\theta$ downwards and the component acceleration $g\sin\alpha$ is also downwards.

Hence the range down the plane

R_2 =distance travelled along the plane in time T

$$= u\cos\theta\,.T + \frac{1}{2}\,g\sin\alpha\,.T^2$$

$$= \frac{2u^2 \sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta + \sin\alpha\ \sin\theta)$$

$$= \frac{2u^2 \sin\theta}{g\cos^2\alpha}\cos(\theta-\alpha) = \frac{u^2}{g\cos^2\alpha}[\sin(2\theta-\alpha)-\sin\alpha]$$

R_2 is maximum, when $\sin(2\theta-\alpha) = 1$.

So maximum range down the plane

$$= \frac{u^2}{g\cos^2\alpha}(1+\sin\alpha) = \frac{u^2}{g(1-\sin\ \alpha)}$$

$$\therefore\ \frac{Max. range\ down\ the\ plane}{Max. range\ up\ the\ plane}$$

$$= \frac{u^2}{g(1-\sin\alpha)}.\frac{g(1+\sin\alpha)}{u^2} = \frac{(1+\sin\alpha)}{(1-\sin\alpha)}$$

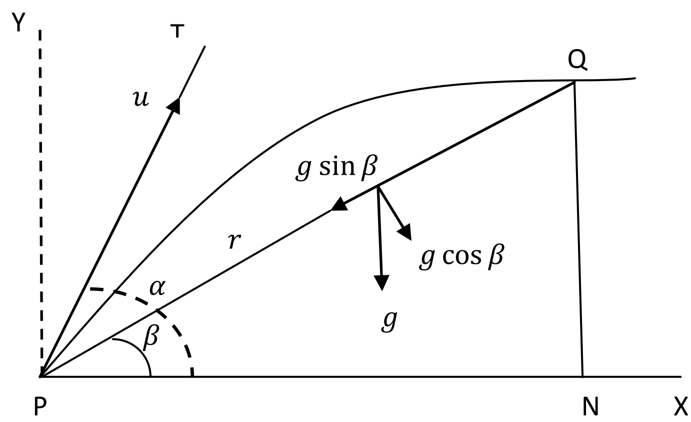
Note. The range R_2 down the plane can be got from the range R_1 up the plane, by changing α into $-\alpha$.

Example 2. A particle is projected at an angle α with a velocity u and it strikes up an inclined plane of inclination β at right angles to the plane. Prove that

$$(i)\cot\beta = 2\tan(\alpha-\beta)$$

$$(ii)\cot\beta = \tan\alpha - 2\tan\beta$$

If the plane is struck horizontally, show that $\tan\alpha = 2\tan\beta$.



Refer to the above figure. The initial velocity $T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$...... (1)

Since the particle strikes the inclined plane normally, its velocity parallel to the inclined plane at the end of time T is $= 0$.

$$0 = u\cos(\alpha-\beta) - g\sin\beta\,.T$$

(or) $T = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$ .... (2)

Equating (1) and (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

i.e. $\cot\beta = 2\tan(\alpha-\beta)$ ...... (i)

i.e. $\cot\beta = \frac{2(\tan\alpha-\tan\beta)}{1+\tan\alpha\tan\beta}$

Cross multiplying,

$$\cot\beta + \tan\alpha = 2\tan\alpha - 2\tan\beta \quad \text{(or)}$$

$$\cot\beta = \tan\alpha - 2\tan\beta \quad .....(ii)$$

If the plane is struck horizontally, the vertical velocity of the projectile at the end of time T is $= 0$. Initial vertical velocity $= u\sin\alpha$, and acceleration in this direction $= g$ downwards.

Vertical velocity in time $T = u\sin\alpha - gT$

$\therefore\ u\sin\alpha - gT = 0$ or $T = u\sin\alpha/g$ ..... (3)

Equating (1) and (3), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\sin\alpha}{g}$$

or $2\sin(\alpha-\beta) = \sin\alpha\cos\beta$

i.e. $2(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin\alpha\cos\beta$.

i.e. $\sin\alpha\cos\beta = 2\cos\alpha\sin\beta$ (or) $\tan\alpha = 2\tan\beta$.

Check Your Progress

1. Define angle of projection
2. Define velocity of projection
3. Describe trajectory and range
4. What is meant by the time of flight of a projectile

10.8 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The **angle of projection** is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.
2. The **velocity of projection** is the velocity with which the particle is projected.
3. The **trajectory** is the path which the particle describes.
4. The **range** on a plane through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.
5. The **time of flight** is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

10.9 SUMMARY

1) The terms are classified as follows:

- The **angle of projection** is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.
- The **velocity of projection** is the velocity with which the particle is projected.
- The **trajectory** is the path which the particle describes.
- The **range** on a plane through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.
- The **time of flight** is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

2) The Greatest height attained by a projectile is $h = \frac{u^2 \sin^2 \alpha}{2g}$.

3) The time taken to reach the greatest height is $T = \frac{u \sin \alpha}{g}$.

10.10 KEYWORDS

Projectile: A projectile is an object upon which the only force acting is gravity.
Examples of a Projectile:

- An object dropped from rest is a projectile (provided that the influence of air resistance is negligible).
- An object that is thrown vertically upward is also a projectile (provided that the influence of air resistance is negligible).
- An object which is thrown upward at an angle to the horizontal is also a projectile (provided that the influence of air resistance is negligible).

10.11 SELF ASSESSMENT QUESTIONS AND EXERCISES

1) If the time of flight of a shot is T seconds over a range of x meters, show that the elevation is $\tan^{-1}\left(\frac{gT^2}{2x}\right)$ and determine the maximum height and the velocity of projection.
2) Show that the greatest height reached by a projectile whose initial velocity is V and angle of projection is α is unaltered if V is increased to kV and α is decreased by λ where $cosec\,\lambda = k(\cot\lambda - \cot\alpha)$.
3) A particle is projected from a point P with a velocity of 32m. per second at an angle of $30°$ with the horizontal. If PQ be its horizontal range and if the angles of elevation from P and Q at any instant of its flight be α and β respectively, show that $\tan\alpha + \tan\beta = \frac{1}{\sqrt{3}}$.
4) A particle is projected from the top of a plane inclined at $60°$ to the horizontal. If the direction of projection is (i) $30°$ above the horizontal and (ii) $30°$ below the horizontal , show that the range down the plane in the first case is double that in the second.
5) A particle is projected with speed u so as to strike at right angles a plane through the point of projection inclined at $30°$ to the horizon. Show that the range on this inclined plane is $\frac{4u^2}{7g}$.
6) If U and V be the oblique components of the initial velocity in the vertical direction and in the direction of the line of greatest slope, show that the range on the inclined plane is $2\,\frac{UV}{g}$.

10.12 FURTHER READINGS

1. Dr. M. K. Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.
2. Dr. M. K. Venkataraman, Dynamics, Agasthiar Publications, 13th Edition, 2009.
3. P. Duraipandian, Laxmi Duraipandian & Muthamizh Jayapragasam, Mechanics,S.Chand&Co.Pvt.Ltd,2014.

UNIT XI IMPULSIVE FORCES

STRUCTURE

11.0 Introduction
11.1 Objectives
11.2 Impulsive Force
11.3 Impact of two bodies
11.4 Loss of Kinetic energy in impact
11.5 Impact of water
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11.0 INTRODUCTION

The term impulse of force is defined as follows:

1) The impulse of a constant force F during a time interval T is defined to be the product FT.

Let f be the constant acceleration produced on a particle of mass m on which F acts and u, v be respectively the velocity at the beginning and end of the period T. Then $v - u = f.T$ and $F = mf$.

Hence the impulse $I = FT = mfT = m(v - u)$

=Change of momentum produced.

2) The impulse of a variable force F during a time interval T is define to be the time
of the force for that interval i.e. impulse $I = \int_0^T F dt$. This is got as follows. During a short interval of time Δt, the force F can be taken to be constant and hence elementary impulse in this interval $= F\Delta t$. Hence the impulse during the whole time T for which the force F acts is the sum of such impulse and

$$= \lim_{\Delta t \to 0} \sum_{t=0}^{T} F.\Delta t = \int_{0}^{T} F dt.$$

Since F is variable, $F = m.\frac{dv}{dt}$.

So impulse $= \int_0^T m\frac{dv}{dt}dt = [mv]_{t=0}^{T} = mv - mu$

Where u and v are the velocities at the beginning and end of the interval and hence this is also equal to the change of momentum produced. Thus whether a force is a variable or constant, its impulse=change of momentum produced.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by impulsive forces
- Discuss the Collision of elastic bodies
- Understand the Newton's experimental law

11.0 IMPULSIVE FORCE

Definition: The change of momentum produced by a variable force P acting on a body of mass $m$from time $t = t_1$ to $t = t_2$ is $\int_{t_1}^{t_2} P\, dt$. Suppose P is very large but the time interval $t_2 - t_1$ during which it acts is very small.

It is quite possible that the above definition integral tends to a finite limit. Such a force is called an *impulsive force*. Thus an impulsive force is one of large magnitude which acts for a very short period of time and yet produces a finite change of momentum.

Theoretically an impulsive force should be infinitely great and the time during which it acts must be very small. This, of course, is never realised in practice, but approximate examples are (1) the force produced by a hammer-blow (2) the impact of a bullet on a target. In such cases the measurement of the magnitude of the actual force is impracticable but the change in momentum produced may be easily measured. Thus an impulsive force is measured by its impulse i.e. the change of momentum it produces.

Since an impulsive force acts only for a short time on a particle, during this time the distance travelled by a particle having a finite velocity is negligible. Also suppose a body is acted upon by impulsive and finite forces simultaneously. Since the time of action *of* the impulsive forces is very short, during this time, the effect of the ordinary finite forces can be neglected.

11.2 IMPACT OF TWO BODIES

If two bodies A and B impinge on each other, then we know by Newton's third law that the action of A on B is equal and opposite to that of B on A, during the period in which they are in contact, and further these forces of action and reaction act along the common normal to the surfaces which are in contact. Hence the impulse of the force exerted by A on B is equal and opposite to that of the force exerted by B on A. It follows that the change in momentum of A is equal and opposite to the change in momentum of B, the momenta being measured along the common normal. Hence taken together, the total change of momenta of A and B is zero. In other words, the sum of the momenta of the bodies, measured along the common normal, is not altered by impact. This is called the $Principle\ of\ conservation\ of\ Linear\ Momentum$ which is used in dealing with problems in which impacts or impulsive forces occur.

11.3 LOSS OF KINETIC ENERGY IN IMPACT

Let a mass m, moving with velocity v, strike a mass M, which is free to move in he direction of $m's$ motion. After impact, let the two move together as a single body with velocity V. As there is no loss of momentum due to the impact,

$$(m+M)V = mv \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The kinetic energy before impact $=\frac{1}{2}mv^2$.

The K. E. after impact $=\frac{1}{2}(m+M)V^2$.

$$= \frac{1}{2}(m+M).\frac{m^2v^2}{(m+M)^2} = \frac{1}{2}m\left(\frac{m}{m+M}\right)v^2.$$

Since, $\left(\frac{m}{m+M}\right)$ is <1, the K. E. after impact is clearly less than $\frac{1}{2}mv^2$ which is K. E. before impact. Hence there is a loss of kinetic energy due to impact and hence the principle of energy must never be used in problems where impulsive forces occur.

11.4 MOTION OF A SHOT AND GUN

When a gun is fired, powder is immediately converted into a gas at a very high pressure and this gas in trying to expand, forces the shot forwards. An equal and opposite reaction is exerted on the gun. The forward momentum generated in the shot at the instant when it leaves the barrel is equal to the backward momentum generated in the gun. Thus, if m and M be the masses of the shot and the gun, v being the muzzle velocity with which the shot emerges from the gun, the gun will recoil with a velocity V given by $MV = mv$.

When the barrel of the gun is elevated, we cannot say that the momenta of the shot and the gun are equal and opposite. In this case, the horizontal momentum of the gun are equal and opposite to the horizontal momentum of the shot. The vertical momentum imparted to the gun will be at once destroyed by the impulsive pressure of the plane on which it stands.

11.5 IMPACT OF THE WATER ON A SURFACE

In the case of a jet of water impinging against a fixed surface or a continuous fall of rain on the ground, we are dealing with a series of successive impacts or impulsive forces. We can calculate the amount of momentum destroyed per second and this will give us the average force on the surface.

11.6 WORKED EXAMPLES

Example 1. *A 100 gm. Cricket ball moving horizontally at 24m/sec was hit straight back with a speed of 15 m/sec. If the contact lasted* $\frac{1}{20}$ *second, find the average force exerted by the bat.*

Let F dynes be the average force exerted by the bat on the ball and I be its impulse.

$I =$ Change of momentum produced.

$= 100[24 - (-15) = 3900]$ units

We know that $I = Ft$

$$\therefore F = \frac{I}{t} = \frac{3900}{\frac{1}{20}} = 78000 \text{ dynes}$$

Example 2. *A jet of water leaves a nozzle of 3cm. diameter at a speed of 2m/sec. and impinges normally on a plane inelastic wall so that the velocity of the water is destroyed on reaching the wall. Calculate in gm. Weight the thrust on the wall.*

Area of cross section of the nozzle$=\pi\left(\frac{3}{2}\right)^2 = 7.07\ cm^2$.

As water issues forthwith a velocity of $200\ cm/sec$, a column of length $200\ cm$. is discharged every second.

$\therefore$ Volume of water discharged per second $= 7.07 \times 200 = 1414 cm^3$.

Since density of water $= 1\ gm/cm^3$, mass of water discharged per sec.$= 1414 gms$. and its velocity is $200\ cm/sec$ which is reduced to zero after striking the wall.

$\therefore$ The momentum destroyed per sec.

$$= 1414 \times 200 = 282800\ units\ (absolute)$$

$\therefore$ Thrust on the wall $= 282800\ dynes$

$$= \frac{282800}{981}\ or\ 288.2\ gms\ wt.$$

Example 3. 8 *centimeters of rain fall in a certain district in* 24 *hours. Assuming that the drops fall freely from a height of* 109 *meters, find the pressure on the ground per square kilometer of the district.*

The velocity of rain on striking the ground

$$= \sqrt{2g \times 109} = \sqrt{2 \times 981 \times 109}$$

$$= \sqrt{2 \times 9 \times 109 \times 109}$$

$$= 327\sqrt{2}\ cm.sec.$$

Volume of rain that falls on a sq. cm in 24 hours

$$= 1^2 \times 8 = 8cc$$

$\therefore$ Mass of rain that falls on a sq. cm. in one sec.

Notes

$$8 \times 1 \times \frac{1}{24 \times 60 \times 60} = \frac{1}{10800} gms.$$

$\therefore$ Momentum destroyed per sec. due to reaction of the ground on the rain drops $= \frac{1}{10800} 327\sqrt{2}\, units.$

The impulsive pressure on the ground is equal to the number of units of momentum destroyed per sec.

$\therefore$ Pressure on the ground per sq. cm. $\frac{327\sqrt{2}}{10800}$ dynes.

Pressure per sq. km $\frac{327\sqrt{2}}{10800} \times 10^{10}$ dynes.

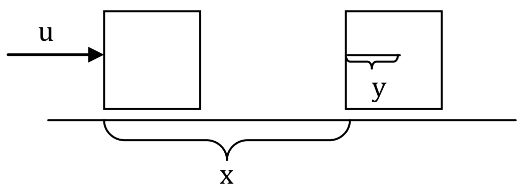
Pressure per sq. km= $\frac{327\sqrt{2}\times 10^{10}}{10800 \times 981} = \frac{\sqrt{2}\times 10^{8}}{324}\; gms.wt.$

Example 4. *A shot of mass m penetrates a thickness t of a fixed plate of mass M. If M were free to move and the resistance supposed to be uniform, show that the thickness penetrated is $\frac{Mt}{M+m}$.*

Let u denote the initial velocity of the shot and F the force of resistance of the plate to the shot which is equal and opposite to that on the plate by the shot. Let f be the retardation due to the resistance. In the first case, when the plate is fixed, the velocity u of the shot is reduced to O by retardation fin a distance t.

$\therefore 0 = u^2 - 2ft$ or $f = \frac{u^2}{2t}$

$\therefore F = mf = \frac{mu^2}{2t}$ ……………………..(1)



In the second case when the plate is free to move, as the shot penetrates the plate, the velocity of the shot diminishes due to the resisting force F and the velocity of the plate increases from zero due to the equal and opposite reaction action on it. The penetration will last as long as the velocity of the shot gun is greater than the velocity of the plate and it will stop when both the plate and the shot acquire a common velocity v (say). Let x be the distance moved by the plate up to this instant and y the thickness penetrated by the shot into the plate. Then $x + y$ is the distance traversed by the shot in space.

By the principle of conservation of momentum for the shot and plate considered as a compound as a compound body, we have

$(M + m)v = mu$ … … … … … … … … (2)

The acceleration due to F on the plate of mass $M = \frac{F}{M} = \frac{mu^2}{2tM}$ using (1). Due to this acceleration, the velocity of the plate increases from 0 to v in a distance x.

$\therefore v^2 = \frac{2mu^2}{2tM}x = \frac{mu^2}{tM}\ x$ or $x = \frac{v^2tM}{mu^2}$ …………….…..(3)

Considering the motion of the shot in space, its velocity decreases from u to v in a distance $x + y$ due to the retardation $f. \therefore v^2 = u^2 - 2f.(x + y) or\ x + y = \frac{u^2-v^2}{2f} = \frac{u^2-v^2}{2\left(\frac{u^2}{2t}\right)} =$

$\frac{t(u^2-v^2)}{u^2}$ ………………… (4)

(4)-(3) gives $y = \frac{t(u^2-v^2)}{u^2} - \frac{v^2tM}{mu^2}$

$$= t - \frac{v^2t}{u^2} - \frac{v^2tM}{u^2m} = t - \frac{v^2t}{mu^4}(m + M)$$

$= t - \frac{t}{mu^2}(m + M)\left(\frac{mu}{m+M}\right)^2$ substituting for v from (2)

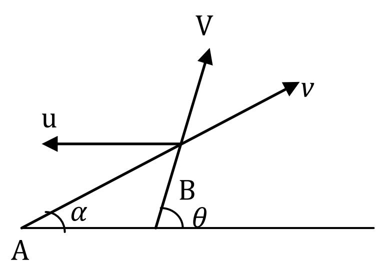
$= t - \frac{tm}{m+M} = \frac{Mt}{m+M}$ and this is the thickness penetrated.

Example 5. *A gun of mass M fires a shell of mass m, the elevation of the gun being α. If the gun can recoil freely in the horizontal direction, show that the angle θ which the path of the shell initially makes with the horizontal is given by the equation $tan\,\theta = \left(1 + \frac{m}{M}\right) tan\,\alpha$.*

Further assuming that the whole energy of the explosion is transferred to the shell and the gun, show that the muzzle energy of the shell is less than what it would be if the gun were fixed, in the ratio $M: M + m\,cos^2\,\theta$.

Let AB represents the barrel of the gun. As the shot leaves the barrel, the gun is moving backwards. Let u be this backward velocity of the gun. Let v be the velocity of the shot relative to the gun. This velocity will be in the direction of the barrel AB.

Let V be the actual velocity of the shot at an angle θ to the horizontal as shown in the figure. We know that V is the resultant of v and u.



Resolving V horizontally and vertically we have

$V\cos\theta = v\cos\alpha - u$ ……………… (1)and $V\sin\theta = v\sin\alpha$ ……………… (2)

Notes

Dividing (2) by (1), $\tan\theta = \frac{v\sin\alpha}{v\cos\alpha - u}$. ……………… (3)

Also the horizontal forward momentum of the shot is = the horizontal backward momentum of the gun *i.e.* $mV\cos\theta = Mu$ ……………… (4)

$i.e. m(v\cos\alpha - u) = Mu$ using (1)

$mv\cos\alpha = (m+M)u$ or $v = \frac{(m+M)u}{m\cos\alpha}$ ……………… (5)

Putting this value of v from (5) in (3), we have

$$\tan\theta = \frac{\frac{(m+M)u}{m\cos\alpha}\sin\alpha}{\frac{(m+M)u}{m\cos\alpha}\cos\alpha - u} = \frac{(m+M)\sin\alpha}{(m+M)\cos\alpha - m\cos\alpha}$$

$$= \frac{(m+M)\sin\alpha}{M\cos\alpha} = \left(\frac{m+M}{M}\right)\tan\alpha = \left(\frac{m}{M}+1\right)\tan\alpha.$$

Energy of explosion = energy of shot + energy of gun

$$= \frac{1}{2}mV^2 + \frac{1}{2}Mu^2.$$

If the gun had been fixed, all the energy of explosion would have been transferred to the shot.

∴ Muzzle energy of the shot when the gun is fixed $= \frac{1}{2}mV^2$.

Required ratio $= \frac{\frac{1}{2}mV^2}{\frac{1}{2}mV^2 + \frac{1}{2}mu^2} = \frac{mV^2}{mV^2+Mu^2}$.

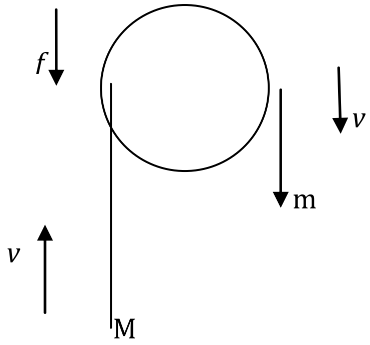
$$= \frac{m.\left(\frac{M^2u^2}{m^2\cos^2\theta}\right)}{m.\left(\frac{M^2u^2}{m^2\cos^2\theta}\right) + Mu^2} \; substuting\ for\ V\ from\ (4)$$

$$= \frac{mM^2u^2}{mM^2u^2 + Mm^2u^2\cos^2\theta} = \frac{M}{M + m\cos^2\theta}.$$

Example 6. *A mass m after falling freely through a distance $'a'$ begins to raise a mass M greater than itself and connected with it by means of an inextensible string passing over a fixed pulley. Show that M will have returned to its original position at the end of time $\frac{2m}{M-m}\sqrt{\frac{2a}{g}}$. Find also what fraction of the kinetic energy of m is destroyed at the instant when M is jerked into motion.*

Velocity of m when it has fallen through a distance ‘a’ $= \sqrt{2ga} = u\ (say)$.

Now the string becomes tight and there is a jerk in the string. This jerk is in the nature of an impulsive force, being a very great force acting for a short time. As a result of this impulsive action, the system acquires a common velocity v and then finite motion begins. M rises upwards with this velocity.



By the principle of conservation of momentum,

$$(M+m)v = mu = m\sqrt{2ga}$$

… … … … … … (1)

Let f be the common acceleration of the system.

We know that $f = \frac{(M-m)g}{M+m}$.

This acceleration will be action on M downwards. M has a velocity v upwards.

So it will rise to a certain height from its first position for a time t given by $0 = v - ft$.

$$i.e. t = \frac{v}{f} = \frac{v(M+m)}{(M-m)g} = \frac{m\sqrt{2ag}}{(M-m)g} \; using \; (1)$$

Subsequently M will trace its path and reach its previous position after a further interval of time t.

∴ Total time taken by M to reach its original position

$$= 2t = \frac{2m\sqrt{2ag}}{(M-m)g} = \frac{2m}{M-m}\sqrt{\frac{2a}{g}}.$$

K. E. of the System before the jerk $= \frac{1}{2}mu^2 = mag$.

K. E. of the system after the jerk

$$= \frac{1}{2}(M+m)v^2 = \frac{1}{2}(M+m).\frac{m^2u^2}{(M+m)^2} = \frac{m^2ag}{M+m}.$$

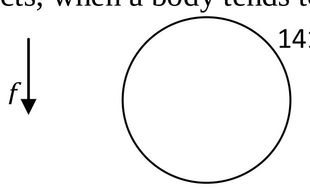
Hence loss of K. E. = $mag - \frac{m^2ag}{M+m} = \frac{mMag}{M+m}$.

∴ Fraction of the K. E. destroyed

$$\frac{mMag}{m+M} \div mag = \frac{M}{m+M}.$$

11.7 COLLISION OF ELASTIC BODIES

A solid body has a definite shape. When a force is applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yields slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force is removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a



Notes

deformation or compression is called the *force of restitution*. Also, the property which causes a solid body to recover its shape is called *elasticity*. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be *inelastic*.

Suppose a ball is dropped from any height h upon a hard floor. It strikes the floor with a velocity $u = \sqrt{2gh}$ and makes an impact. Soon it rebounds and moves vertically upwards with a velocity v. The height h_1 to which it rebounds is given by $h_1 = \frac{v^2}{2g}$ i.e. $v = \sqrt{2gh_1}$. Generally we find that $h_1 < h$. So $v < u$ As soon as the ball strikes the floor, the impulsive action of the floor rapidly stops the downward velocity of the ball and at the same time causes a temporary compression near the point of contact. Due to the elastic property of the solid, the ball tends to regain its original form quickly. It presses the floor and receives an equal and opposite impulsive reaction from it and with a new upward velocity, it rebounds.

(like wooden floor, marble floor etc), it will be found that the heights to which they rebound after striking the floor will be different. In all these cases, the velocity of the ball on reaching the floor is the same, as it is dropped from the same height. But the velocity of the ball after impact is not the same in each case, as the height to which it rebounds is different. Thus due to the elastic property of solid bodies, a change in velocity takes place when the strike each other.

If $v = u$, the velocity with which the ball leaves the floor is the same as that with which it strikes it. In this case, the ball is said to be *perfectly elastic*. If $v = 0$, the ball does not rebound at all. It is said to be *inelastic*. More generally, when a body completely regains its shape after a collision, it is said to be $perfectly\,elastic$. If it does not come to its original shape, it is said to be perfectly inelastic. These two case of bodies are only ideal.

In this chapter, we shall study some simple cases of the impact of elastic bodies. We shall consider the cases of particles in collision with particles, or planes and of spheres in collision with planes or spheres. In all cases, we consider the impinging bodies to be smooth, so that the only mutual action they can have on each other will be along the common normal at the point where they touch.

11.8 DEFINITIONS

Two bodies are said to *impinge directly* when the direction of motion of each before impact is along the common normal at the point where they touch.

They are said to *impinge obliquely*, if the direction of motion of either body or both is not along the common normal at the point where they touch.

The common normal at the point of contact is called the *line of impact*. Thus, in the case of two spheres, the line of impact is the line joining their centers.

11.9 FUNDAMENTAL LAWS OF IMPACT

The following three general principles hold good when two smooth moving bodies make an impact.

11.10 NEWTON'S EXPERIMENTAL LAW

Newton studied the rebound of elastic bodies experimentally the result of his experiments is embodied in the following law.

When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e, and is called the *coefficient (or modulus) of elasticity (or restitution or resilience).*

This law can be put symbolically as follows: If u_1, u_2 are the components of the velocities of two impinging bodies along their common normal before impact and v_1, v_2 their component velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution, then

$$\frac{v_2 - v_1}{u_2 - u_1} = -e.$$

The quantity e, which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls it is about 0.9; for ivory 0.8; while for lead it is 0.2. For two balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which $e = 0$ are said to be inelastic while for perfectly elastic bodies, $e = 1$. Probably, there are no bodies in nature coming strictly under either of these headings. Newton's law is purely empirical and is true only approximately, like many experimental laws.

11.11 MOTION OF TWO SMOOTH BODIES PERPENDICULAR TO THE LINE OF IMPACT

When two smooth bodies impinge, the only force het them at the time of impact is the mutual reaction which acts the common normal. There is no force acting along the common tangent and hence there is no change of velocity in that direction. Hence the *velocity of either body resolved in a direction perpendicular to the line of impact is not altered by impact.*

11.12 PRINCIPLE OF CONSERVATION OF MOMENTUM

We can apply the law of conservation of momentum in the case of two impinging bodies. *The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum their momenta before impact, all momenta being measured along the common normal.*

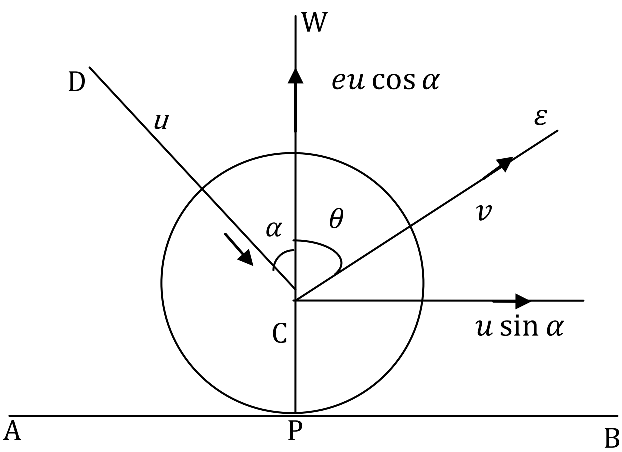
The above three principles are sufficient to study the changes in the motion of two impinging elastic bodies.
We shall now proceed to discuss particular cases.

11.13 IMPACT OF A SMOOTH SPHERE ON A FIXED SMOOTH PLANE

A smooth sphere, or particle whose mass is m and whose coefficient of restitution is e, impinges obliquely on a smooth fixed plane; to find its velocity and direction of motion after impact.

Let AB be the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the center of the sphere. Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure.

Since the plane and the sphere are smooth, the only force acting during impact is



the impulsive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence the velocity of the sphere, resolved in a direction parallel to the plane is unaltered by the impact.

Hence $v\sin\theta = u\sin\alpha$ ……………… (1)

By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is $(-e)$ time its relative velocity along the common normal before impact. Hence

$$v\cos\theta - 0 = -e(-u\cos\alpha - 0)$$

$i.e. v\cos\theta = e\,u\cos\alpha$ ……………… (2)

Squaring (1) and (2), and adding, we have

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \quad i.e.\, v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Dividing (2) by (1), we have $\cot\theta = e\cot\alpha$ ……………… (4)

Hence (3) and (4) give the velocity and direction of motion after impact.

Corollary 1. If $e = 1$, we find that from (3) $v = u$ and from (4) $\theta = \alpha$ Hence is a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Corollary 2. If $e = 0$, then from (2), $v\cos\theta = 0$ and from (3), $v = u\sin\alpha$. Hence $\cos\theta = 0$ i.e. $\theta = 90^o$. Hence the inelastic sphere slides along the plane with velocity $u\sin\alpha$.

Corollary 3. If the impact is direct we have $\alpha = 0$. Then $\theta = 0$ and from (3) $v = eu$. Hence if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu.

Corollary 4. The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$$I = mv\cos\theta - (-mu\cos\alpha) = m(v\cos\theta + u\cos\alpha)$$
$$= m(eu\cos\alpha + u\cos\alpha) = mu\cos\alpha\,(1+e)$$

Corollary 5. Loss of kinetic energy due to the impact

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}mu^2(\sin^2\alpha + e^2\cos^2\alpha)$$
$$= \frac{1}{2}mu^2(1 - \sin^2\alpha - e^2\cos^2\alpha) = \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$$
$$= \frac{1}{2}(1-e^2)mu^2\cos^2\alpha$$

If the sphere is perfectly elastic $e = 1$ and the loss of kinetic energy is zero.

11.14 WORKED EXAMPLES

Example 1. *Smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball projected along the table from a point A on the rim in a direction making an angle α with the radius through A will return to the point of projection after two impacts if $tan\,\alpha = \frac{e^{\left(\frac{3}{2}\right)}}{\sqrt{1+e+e^2}}$. Prove also that when the ball returns to the points of projection, its velocity is to its original velocity as* $(3/2){:}\,1$.

Let the ball starting from A return to it after two reflection at B and C. At B, the point of the first impact, the common normal is the radius OB and at C, the point of the second impact, the common normal is OC.

Let $\angle OBC = \beta$ and $\angle OCA = \gamma$. Then $\angle OCB = \beta$ and $\angle OAC = \gamma$.

Notes

Considering the impact at B, and applying equation (4) of previous section, we have $\cot\beta = e\cot\alpha$. i.e. $\tan\beta = \frac{1}{e}$ ... ... (1)

Similarly, considering the impact at C, $\cot\gamma = e\cot\beta$.

$\tan\gamma = \frac{1}{e}\tan\beta = \frac{1}{e^2}\tan\alpha.$ ... ... (2)

Now, in ΔABC, $\angle A + \angle B + \angle C = 2\alpha + 2\beta + 2\gamma = 180^o$.

$\therefore \alpha + \beta + \gamma = 90^o$ or $\alpha = 90^o - (\beta + \gamma)$.

$$\therefore \tan\alpha = \tan 90^o - \overline{(\beta+\gamma)} = \cot(\beta+\gamma)$$
$$= \frac{1}{\tan(\beta+\gamma)} = \frac{1 - \tan\beta\tan\gamma}{\tan\beta + \tan\gamma}.$$

i.e. $\tan\alpha(\tan\beta + \tan\gamma) = 1 - \tan\beta\tan\gamma$.

i.e. $\tan^2\alpha\left(\frac{1}{e} + \frac{1}{e^2}\right) = 1 - \frac{tan^2\alpha}{e^3}$.

$\tan^2\alpha\left(\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3}\right) = 1$ or $\tan^2\alpha\left(\frac{1+e^2+e}{e^3}\right) = 1$.

$i.e.\quad \tan^2\alpha = \frac{e^3}{1+e^2+e\ldots}$ ... ... (3)

Or $\tan\alpha = \frac{e^{(\frac{3}{2})}}{\sqrt{1+e^2+e}}$.

Let u be the velocity of projection from A, v be the velocity of the ball after the first impact at B and w be the velocity after the second impact at C.

Applying equation (3) of section 11.13, we have

$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$ and $w^2 = v^2(\sin^2\beta + e^2\cos^2\beta)$

$$w^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)(\sin^2\beta + e^2\cos^2\beta)$$
$$= u^2\cos^2\alpha(\tan^2\alpha + e^2).\cos^2\beta(\tan^2\beta + e^2)$$
$$= \frac{u^2(\tan^2\alpha + e^2)(\tan^2\beta + e^2)}{(\tan^2\alpha + 1)(\tan^2\beta + 1)}.$$
$$= \frac{u^2(\tan^2\alpha + e^2)\left(\frac{1}{e^2}\tan^2\alpha + e^2\right)}{(\tan^2\alpha + 1)\left(\frac{1}{e^2}\tan^2\alpha + 1\right)}. \qquad using (1)$$
$$= \frac{u^2(\tan^2\alpha + e^2)(\tan^2\alpha + e^4)}{(\tan^2\alpha + 1)(\tan^2\alpha + e^2)} = \frac{u^2(\tan^2\alpha + e^4)}{(\tan^2\alpha + 1)}$$
$$= \frac{u^2\left[\frac{e^4}{1+e+e^2} + e^4\right]}{\left[1 + \frac{e^4}{1+e+e^2}\right]} \quad substituting\ from\ (3)$$
$$= \frac{u^2(e^3 + e^4 + e^5 + e^6)}{(1 + e + e^2 + e^3)} = u^2e^3$$

$\therefore w = u.e^{(\frac{3}{2})}$ or $w : u = e^{(\frac{3}{2})} : 1$.

Example 2. *A particle falls from a height h upon a fixed horizontal plane: if e be the coefficient of restitution, show that the whole distance described before the* *particle has finished rebounding is* $h\left(\frac{1+e^2}{1-e^2}\right)$. *Show also that the whole time taken is* $\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$.

Let u the velocity of the particle on first hitting the plane. Then $u^2 = 2gh$. After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane with velocity e^2u and the process is repeated a number of times. The velocities after the third, fourth etc. impacts are e^3u, e^4u etc.

The height ascended after the first impact with velocity eu=$\frac{(velocity\ )^2}{2g} = \frac{e^2u^2}{2g}$.

The height ascended after the second impact with velocity $e^2u = e^4u^2/2g$ and so on.

∴ Total distance travelled before the particle stops rebounding

$$h + 2\left(\frac{e^2u^2}{2g} + \frac{e^2u^2}{2g} + \frac{e^2u^2}{2g} + \cdots\right)$$

$$= h + \frac{2.e^2u^2}{2g}(1 + e^2 + e^4 + \cdots to\ \infty)$$

$$= h + \frac{e^2u^2}{g}.\frac{1}{1-e^2} = h + \frac{e^2 2gh}{h}.\frac{1}{1-e^2}$$

$$= h + \left(\frac{2e^2}{1-e^2}\right) = h.\left(\frac{1+e^2}{1-e^2}\right)$$

Consider the motion before the first impact, we have the initial velocity= 0, acceleration= g, final velocity = u and so if t is the time taken, $u = 0 + gt$.

$$\therefore t = \frac{u}{g} = \frac{velocity}{g}.$$

Time interval between the first and second impact is

= 2 × time taken for gravity to reduce the velocity eu to 0.

= 2. velocity $/g = 2eu/g$.

Similarly time interval between the second and third impacts= $2e^2u/g$ and so on.

So total time taken $= \frac{u}{g} + 2\left(\frac{eu}{g} + \frac{e^2u}{g} + \cdots\infty\right)$

$$= \frac{u}{g} + \frac{2eu}{g}(e + e^2 + \cdots\infty)$$

$$= \frac{u}{g} + \frac{2eu}{g}.\frac{1}{1-e} = \frac{u}{g}\left[1 + \frac{2e}{1-e}\right].$$

$$= \frac{u}{g}\left(\frac{1+e}{1-e}\right) = \frac{\sqrt{2gh}}{g}\left(\frac{1+e}{1-e}\right) = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}.$$

Example 3. A ball is thrown from a point on a smooth horizontal ground with a speed V at an angle α to the horizon. If e be the coefficient of restitution, show that the total time for which the ball rebounds on the ground is $\frac{2V\sin\alpha}{g(1-e)}$ and the horizontal distance travelled by it is $\frac{V^2\sin 2\alpha}{g(1-e)}$.

The initial horizontal and vertical components of the velocity are $V\cos\alpha$ and $V\sin\alpha$. The particle describes a parabola and strikes the horizontal velocity is not affected while the vertical component is reversed as $eV\sin\alpha$. Similarly the vertical components of the velocity after the second, third etc., impacts are $e^2V\sin\alpha, e^3V\sin\alpha$ etc.

Let t_1, t_2, t_3 etc be the times for the successive trajectories.

$t_1 = \frac{2V\sin\alpha}{g}, t_2 = \frac{2eV\sin\alpha}{g}, t_3 = \frac{2e^2V\sin\alpha}{g}$ and so on.

So total time that elapses before the particle stops rebounding

$$= \frac{2V\sin\alpha}{g} + \frac{2eV\sin\alpha}{g} + \frac{2e^2V\sin\alpha}{g} + \cdots$$

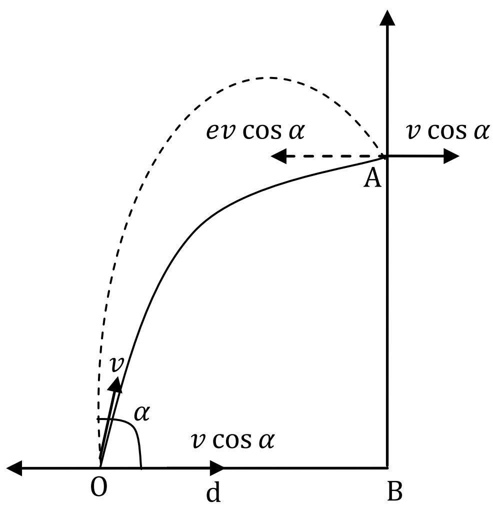
$$= \frac{2V\sin\alpha}{g}(1 + e + e^2 + \cdots to\ \infty) = \frac{2V\sin\alpha}{g}.\frac{1}{1-e}$$

$= \frac{2V\sin\alpha}{g(1-e)}.$

Examples 4. *An elastic sphere is projected from a given point O with given velocity V at an inclination α to the horizontal and after hitting a smooth vertical wall at a distance d from O returns to O. Prove that $d = \frac{v^2\sin 2\alpha}{g}\frac{e}{1+e}$ where e is the coefficients of restitution.*

Let the particle strike the wall at A. From O to A, the particle describes a parabola under gravity with constant horizontal velocity $v\cos\alpha$. Let t_1 be the time for this,

$\therefore\ v\cos\alpha . t_1 = d$ ... ... (1)



At the impact at A, there is no force parallel to the wall. The component $v\cos\alpha$ being perpendicular to the wall is reversed as $ev\cos\alpha$. The particle will describe another parabola with constant horizontal velocity $ev\cos\alpha$ and return to O. Let t_2 be the time for this return journey. Then $ev\cos\alpha . t_2 = d$ ... ... (2)

But the vertical motion is not affected by impact and throughout the interval $t_1 + t_2$, it is subject to retardation by g only. As the particle returns to O, vertical distance described in time $t_1 + t_2 = 0$.

$$\therefore 0 = v \sin\alpha\,(t_1 + t_2) - \frac{1}{2} g (t_1 + t_2)^2 \quad \text{or}$$

$$t_1 + t_2 = \frac{2v \sin\alpha}{g}$$

Substituting for $t_1 t_2$ from (1) and (2) in (3), we have

$\frac{d}{v\cos\alpha} + \frac{d}{ev\cos\alpha} = \frac{2v\sin\alpha}{g}$ i.e., $\frac{d(e+1)}{ev\cos\alpha} = \frac{2v\sin\alpha}{g}$

Or

$$d = \frac{2ev^2 \sin\alpha\cos\alpha}{g(1+e)} = \frac{v^2 \sin 2\alpha}{g} \cdot \frac{e}{1+e}$$

Check your Progress

1. Define impulse of force.
2. Describe the impact of water on a surface
3. Define the force of restitution
4. What is meant by elasticity and inelasticity

11.15 ANSWERS TO CHECK YOUR PROGRESS QUESTION

1. The change of momentum produced by a variable force P acting on a body of mass $m$from time $t = t_1$ to $t = t_2$ is $\int_{t_1}^{t_2} P\,dt$. Suppose P is very large but the time interval $t_2 - t_1$ during which it acts is very small. It is quite possible that the above definition integral tends to a finite limit. Such a force is called an ***impulsive force***.

2. In the case of a jet of water impinging against a fixed surface or a continuous fall of rain on the ground, we are dealing with a series of successive impacts or impulsive forces. We can calculate the amount of momentum destroyed per second and this will give us the average force on the surface.
3. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the ***force of restitution.***
4. The property which causes a solid body to recover its shape is called ***elasticity***. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be ***inelastic.***

11.16 SUMMARY

- **Impulsive force:** The change of momentum produced by a variable force P acting on a body of mass $m$from time $t = t_1$ to $t = t_2$ is $\int_{t_1}^{t_2} P\,dt$. Suppose P is very large but the time interval $t_2 - t_1$ during which it acts is very small. It is quite possible that the above definition integral tends to a finite limit. Such a force is called an $impulsive\ force$.
- **Principle of conservation of linear momentum:** The sum of the momenta of the bodies, measured along the common normal, is not altered by impact. This is

called the $Principle\,of\ conservation\ of\ Linear\,Momentum$ which is used in dealing with problems in which impacts or impulsive forces occur.

- **Force of restitution:** The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the *force of restitution*
- **Elasticity:** The property which causes a solid body to recover its shape is called *elasticity.*
- **Inelasticity:** . If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be *inelastic*.
- **Perfectly elastic and Perfectly inelastic:** If $v = u$, the velocity with which the ball leaves the floor is the same as that with which it strikes it. In this case, the ball is said to be *perfectly elastic*. If $v = 0$, the ball does not rebound at all. It is said to be *inelastic*. More generally, when a body completely regains its shape after a collision, it is said to be $perfectly\,elastic$. If it does not come to its original shape, it is said to be perfectly inelastic.

- **Impinge directly:** Two bodies are said to *impinge directly* when the direction of motion of each before impact is along the common normal at the point where they touch.
- **Impinge obliquely:** They are said to *impinge obliquely*, if the direction of motion of either body or both is not along the common normal at the point where they touch.
- **Line of impact:** The common normal at the point of contact is called the *line of impact*. Thus, in the case of two spheres, the line of impact is the line joining their centers.

Newton's Experimental law: When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.

Principle of conservation of momentum: The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum their momenta before impact, all momenta being measured along the common normal.

11.7 KEYWORDS

- **Impulsive force:** The change of momentum produced by a variable force P acting on a body of mass $m$from time $t = t_1$ to $t = t_2$ is $\int_{t_1}^{t_2} P\,dt$. Suppose P is very large but the time interval $t_2 - t_1$ during which it acts is very small. It is quite possible that the above definition integral tends to a finite limit. Such a force is called an $impulsive\ force$.
- **Principle of conservation of linear momentum:** The sum of the momenta of the bodies, measured along the common normal, is not altered by impact. This is called the $Principle\,of\ conservation\ of\ Linear\,Momentum$ which is used in dealing with problems in which impacts or impulsive forces occur.

- **Newton's Experimental law:** When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.
- **Elasticity:** The property which causes a solid body to recover its shape is called *elasticity*.

11.18 SELF ASSESSMENT QUESTIONS AND EXERCISES

1) A shell explodes and breaks into two fragments of masses m_1, m_2 moving with initial velocities u_1 and u_2 respectively in opposite directions. Show that there is a gain in kinetic energy of magnitude $\frac{1}{2}\ \frac{m_1\, m_2}{m_1+m_2}(u_1+u_2)^2$.
2) A body of mass m_1+m_2 is split into two parts of masses m_1 and m_2 by an internal explosion which generates a K.E. Show that, if after explosion, the parts move in the same line as before, their relative speed is $\sqrt{\frac{2\,E(m_1+m_2)}{m_1 m_2}}$.
3) A gun of mass M gm firing a shot of mass m gm. recoils with velocity v m. per second. Show that , if the mass of the shot is increased to 2m, the kinetic energy of the explosion remaining the same, the velocity of recoil becomes $v\sqrt{\frac{2(m+m)}{M+2m}}$.
4) A gun of mass M fires a shell of mass m horizontally and the energy of explosion is such as sufficient to project the shell vertically to a height h. Show that the velocity of recoil is $\left[\frac{2\,m^2gh}{M(M+m)}\right]^{\frac{1}{2}}$.
5) A particle falls from a height h in time $t$upon a fixed horizontal plane. Prove that it rebounds and reaches a maximum height e^2h in time et.
6) A heavy ball drops from the ceiling of a room and after rebounding twice from the floor reaches a height equal to one half that of the ceiling. Show that the coefficient of restitution is $\left(\frac{1}{2}\right)^{\frac{1}{4}}$.
7) A ball is projected with a velocity of $24\sqrt{3}$ ft per second at an elevation of $45°$. It strikes a wall at a distance of 18 ft. and returns to the point of projection. Show that $e=\frac{1}{2}$.
8) A ball falls vertically for 2 seconds and hits a plane inclined at $30°$ to the horizon. If the coefficient of restitution be $\frac{3}{4}$, show that the time that elapses befpre it again hits the plane is 3 seconds.
9) A ball is thrown from a point on a smooth horizontal ground with a speed V at an angle α to the horizon. If e be the coefficient of restitution, show that the total time for which the ball rebounds on the ground is $\frac{2V\sin\alpha}{g(1-e)}$ and the horizontal distance travelled by it is $\frac{V^2\sin 2\alpha}{g(1-e)}$.

11.19 FURTHER READINGS

1. Dr. M. K. Venkataraman, Statics, Agasthiar Publications, 17th Edition, 2014.
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BLOCK - IV

UNIT-XII IMPACT OF SPHERE

STRUCTURE

12.0 Introduction
12.1 Objectives
12.2 Direct Impact of Two Smooth Spheres
12.3 Loss of Kinetic Energy due to Direct Impact of Two Smooth Spheres
12.4 Worked examples
12.5 Oblique Impact of Two Smooth Spheres
12.6 Loss of Kinetic Energy due to Oblique Impact of Two Smooth Spheres
12.7 Check your Progress
12.8 Answers to Check Your Progress Questions
12.9 Summary
12.10 Keywords
12.11 Self Assessment Questions and Exercises
12.12 Further Readings

12.0 INTRODUCTION

Impact of two bodies:

If two bodies A and B impinge on each other, then we know by Newton's third law that the action of A and B is equal and opposite to that of B on A, during the period in which they are in contact, and further these forces of action and reaction act along the common normal to the surfaces. Which are in contact? Hence the impulse of the force exerted by A on B is equal and opposite to that of the force exerted by B on A. it follows that the change in momentum of A is equal and opposite to the change in momentum of B, the moments being measured along the common normal. The sum of the moments of the bodies; measured along the common normal is altered by impact. This is called the principle of conservation of linear momentum.

12.1 OBJECTIVES

After going through this unit, you will be able to:

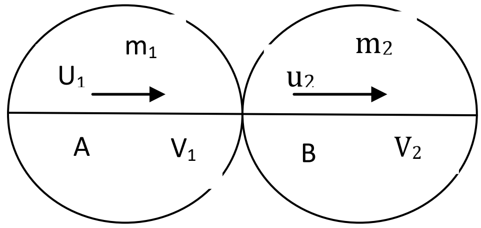
- To find the velocity of the direct impact spheres
- To find the amount of loss K.E due to direct Impact
- To find the velocity of the Oblique Impact of two spheres

12.2 IMPACT OF TWO SMOOTH SPHERES:

A smooth sphere of mass m_1 impinges directly with velocity u_1 on another smooth sphere of mass m_2 moving in the same direction with velocity u_2; if the coefficient of restitution is e, to find their velocities after the impact:

Notes

AB is the line of impact, i.e the common normal. Due to the impact there is no tangential force and hence, for either sphere the velocity along the tangent is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere tangential velocity after impact = its tangent velocity before impact = 0. So, after impact, the spheres will move only in the direction AB. let their velocities be v_1 and v_2.



By Newton's experimental law, the relative velocity of m_2 with respect m_1 after impact is (-e) times the corresponding relative velocity before impact.

$$v_2 - v_1 = -e(u_2 - u_1) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

By the principle of conservation of momentum, the total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

(2) – (1) × m_2 gives

$$v_1(m_1 + m_2) = m_1 u_1 + m_2 u_2 + e m_2(u_2 - u_1)$$

$$= m_2 u_2(1 + e) + (m_1 - e m_2)u_1$$

$$v_1 = \frac{m_2 u_2(1+e)+(m_1-e m_2)u_1}{m_1+m_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

(1)× m_1 + (2) gives

$$v_2(m_1 + m_2) = -e m_1(u_2 - u_1) + m_1 u_1 + m_2 u_2$$

$$= m_1 u_1(1 + e) + (m_2 - e m_1)u_2$$

$$= m_1 u_1(1 + e) + (m_2 - e m_1)u_2$$

Therefore,

$$v_2 = \frac{m_1 u_1(1+e)+(m_2-e m_1)u_2}{m_1+m_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Equations (3) and (4) give the velocities of the spheres after impact

Note: If one sphere says m_2 moving originally in a direction opposite to that of m_1, the sign of u2 will be negative. Also it is important that the directions of v_1 and v_2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v_1 and v_2 are in this direction. If either of them is actually in the opposite direction, the value obtained for it will turn to be negative

In writing equation (1) corresponding to Newton's law, the velocities must be subtracted in the same order on both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies

Corollary 1. If the two spheres are perfectly elastic and of equal mass, e = 1 and $m_1 = m_2$. Then, from equations (3) and (4), we have

$v_1 = \frac{m_1 u_2 .2+0}{2m_1} = u_2$ and $v_2 = \frac{m_1 u_1 .2+0}{2m_1} = u_1$ i.e. If two equal perfectly elastic spheres impinge directly, they interchange their velocities.

Corollary 2. The impulse of the blow on the spheres A of mass m_1= change of momentum of A = $m_1(v_1 - u_1)$.

$$= m_1 \left[\frac{m_2 u_2(1+e)+(m_1-em_2)u_1}{m_1+m_2} - u_1\right]$$

$$= m_1 \left[\frac{m_2 u_2(1+e)+m_1 u_1-em_2 u_1-m_1 u_1-m_2 u_1}{m_1+m_2}\right]$$

$$= m_1 \left[\frac{m_2 u_2(1+e)-m_2 u_1(1+e)}{m_1+m_2}\right]$$

$$= \frac{m_1 m_2(1+e)(u_2-u_1)}{m_1+m_2}$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

12.3 LOSS OF KINETIC ENERGY DUE TO DIRECT IMPACT OF TWO SMOOTH SPHERES:

Two spheres of given masses with given velocities impinge directly; to show that there is a loss of kinetic energy and to find the amount:

Let $m_1 m_2$ be the masses of the spheres, u_1 and u_2 , v_1 and v_2 be their velocities before and after impact and e the coefficient of restitution.

By the Newton's law,

$v_2 - v_1 = -e(u_2 - u_1)$ …………….. (1)

By the principle of conservation of momentum,

$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$ ….. (2)

Notes

Total kinetic energy before

Impact $= \frac{1}{2}\ m_1{u^2}_1 + \frac{1}{2}\ m_2{u^2}_2$

Total kinetic energy after

Impact $= \frac{1}{2}\ m_1{v^2}_1 + \frac{1}{2}\ m_2{v^2}_2$

Change in K.E = initial K.E – final K.E

$$= \frac{1}{2}\ m_1{u^2}_1 + \frac{1}{2}\ m_2{u^2}_2 - \frac{1}{2}\ m_1{v^2}_1 - \frac{1}{2}\ m_2{v^2}_2$$

$$= \frac{1}{2}\ m_1(u_1 - v_1)(u_1 + v_1) + \frac{1}{2}\ m_2(u_2 - v_2)(u_2 + v_2)$$
$$= \frac{1}{2}\ m_1(u_1 - v_1)(u_1 + v_1) + \frac{1}{2}\ m_1\ (v_1 - u_1)(u_2 + v_2)$$
$$= \frac{1}{2}\ m_1(u_1 - v_1)[u_1 + v_1 - (u_2 + v_2)]$$
$$= \frac{1}{2}\ m_1(u_1 - v_1)[u_1 - u_2 - (v_2 - v_1)]$$
$$= \frac{1}{2}\ m_1(u_1 - v_1)[u_1 - u_2 + e(u_2 - u_1)$$ Using(1)
$$= \frac{1}{2}\ m_1(u_1 - v_1)(u_1 - u_2)(1 - e)\ \text{……...}\ (3)$$

Now, from (2)

$$m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

$\frac{u_1 - v_1}{m_2} = \frac{v_2 - u_2}{m_1}$ And each $= \frac{u_1 - v_1 + v_2 - u_2}{m_1 + m_2}$

i.e. each $= \frac{(u_1 - u_2) + (v_2 - v_1)}{m_1 + m_2}$

$= \frac{(u_1 - u_2) + e(u_2 - u_1)}{m_1 + m_2}$ Using (1)

$= \frac{(u_1 - u_2)(1 + e)}{m_1 + m_2}$

$$u_1 - v_1 = \frac{m_2(u_1 - u_2)(1 + e)}{m_1 + m_2}$$

And substituting this in (3),

Change in K.E.

$$= \frac{1}{2}\ \frac{m_1 m_2 (u_1 - u_2)(1 + e)(u_1 - u_2)(1 - e)}{m_1 + m_2}$$

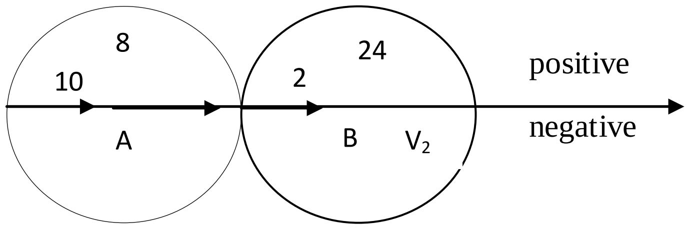
$$= \frac{1}{2}\ \frac{m_1 m_2((u_1 - u_2)^2(1 - e^2)}{(m_1 + m_2)}\quad \text{…….}\ (4)$$

As e<1,the expression (4) is always positive and so the initial kinetic energy of the system is greater than final kinetic energy. So there is a actual loss of total kinetic energy by a collision. Only in the case of, when e = 1, i.e. only when the bodies are perfectly elastic, the expression (4) becomes zero and hence the total kinetic energy is unchanged by impact.

12.4 WORKED EXAMPLES

Example 1: A ball of mass 13 gm. moving with a velocity of 13 cm. per sec. impinges directly on another of mass 24 gm., moving at 2cm/sec in the direction. If e = ½, find the velocities after impact. Also calculate the loss in kinetic energy.

Solution:



Let v_1 and v_2 cm/sec be the velocities of the masses 8gm and 24 gm respectively after impact.

By the Newton's law,

$$24v2+8v1=24\times 2+13\times 13 = 128$$

$$\text{i.e. } 3v2+ v1 = 16 \qquad \text{……….(2)}$$

Solving (1) and (2), we get

$$v_1=1\text{cm, /sec., } v_2= 5\text{cm. /sec}$$

The K.E. before impact $= ½ .13.10^2+ ½ .24. 2^2$

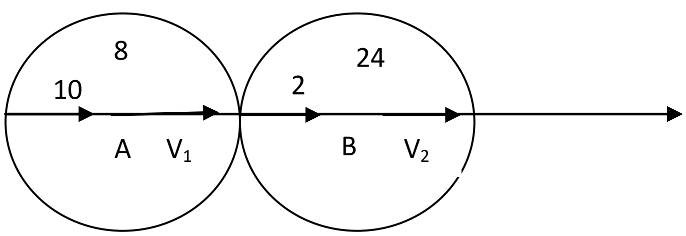
$= 448$ dynes

The K.E. after impact $= ½ .13.1^2+ ½ .24. 5^2$

$= 304$ dynes

Loss in K.E. after impact = 144 dynes.

Example2: If the 24 gm. Mass in the previous question be moving in a direction opposite to that of the 8gm.mass, find the velocities after impact.



Notes

Let v_1 and v_2 cm/sec be the velocities of the 8gm and 24 gm mass respectively after impact.

By the Newton's law,

$$v_2 - v_1 = -\frac{1}{2}(-2 - 13) = 6$$

By the conservation of momentum

$24v_2+8v_1= 24 \times (-2) +13\times13 = 32$

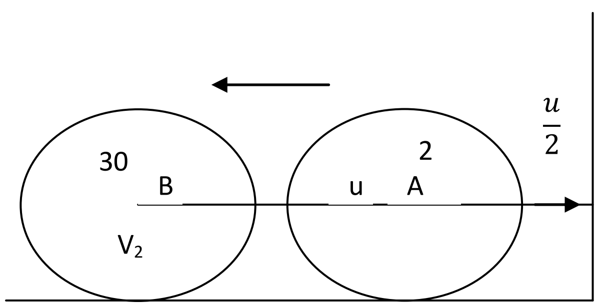
i.e. $3v_2+ v_1 = 4$

Solving (1) and (2),

$v_1= -\frac{1}{2}$ cm/sec., $v_2=\frac{5}{2}$ cm/sec

The negative sign of v_1 shoes that the direction of motion of the 8gm. Mass is reversed ,as we had taken the direction left to right as positive and assumed v_1 to be in this direction. Since v_2 is positive, the 24gm. Ball moves from left to right after impact, so that its direction of motion is also reversed.

Example 3: Two equal spheres A and B, of masses 2 gm. and 30gm. respectively lie on a smooth floor, so that their line of centers is perpendicular to fixed vertical wall. A being nearer to the wall. A is projected toward B. show that if the coefficient of restitution between the two spheres and that between the first sphere and the wall is 3/5 , then A will be reduced to rest after its second impact with B.



Consider the impact between A and B. taking AB as the positive direction, let the velocity of A before impact be u. B is at rest. After theimpact, let the velocities of A and B be v_1 and v_2 respectively in the same direction. By the Newton's law

$$v_2 - v_1 = -e(0 - u) = \frac{3}{5}u$$

By the conservation of momentum, along AB,

$30v_2+2v_1= 30\times0+ 2 \times u$

i.e. $15v_2+ v_1 = u$

Solving (1) and (2),

$$v_1= -\frac{u}{2} \quad v_2= \frac{u}{13}$$

Since v_1 is negative, the velocity of A after the impact towards the wall and = u/2 while the velocity of B is u/13 away from the wall.

Now A strikes the wall with a velocity u/2. After this impact will be reversed as e.

(U/2)= 3/5.U/2=3U/13

With this velocity, A moves in the direction AB, away from the wall and strikes B a second time. Let the velocities of A and B be v_3 and v_4 after this impact, in the direction AB. For convenience, the velocity distribution can be noted as follows.

	A (2)	B (30)
before impact	$\frac{3u}{13}$	$\frac{u}{13}$
after impact	v_3	v_4

By Newton's law,

$v_4 - v_3 = -e\left(\frac{u}{13} - \frac{3u}{13}\right) = \frac{3u}{25}$ ……… (3)

By conservation of momentum,

$30v_4+2v_3 = = 30.\frac{u}{13} + 2.\frac{3u}{13} = \frac{18u}{5}$

i.e. $15 v_4 + v_3 = \frac{9u}{5}$ …………….. (4)

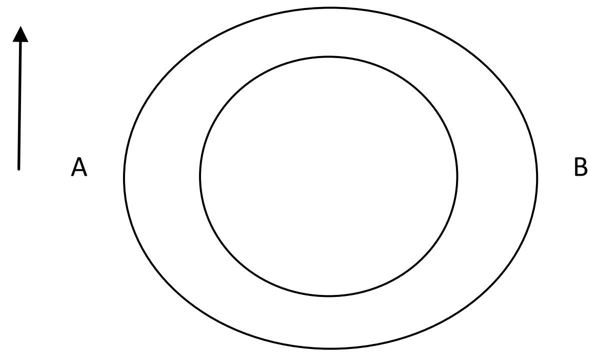
Multiplying (3) by 15, we have

$15 v_4 + 15 v_3 = \frac{9u}{5}$ ……………. (5)

Subtracting (5) from (4), $16 v_3 = 0$ or $v_3 = 0$.

i.e. A is reduced to rest after its second impact with B.

Example 4: Two equal marble balls A, B lie in a horizontal circular groove at the opposite ends of a diameter; A is projected along the groove and after



time t, impinges on B; show that a second impact takes place after a further interval $\frac{2t}{e}$.

Let the ball A move with velocity u. as there is no tangential force acting on A at any point of its path, its speed remains the same throughout. Hence it impinges on B with a velocity u.

Since the time from A to B is = t,

We get $\mathrm{ut} = \pi\mathrm{r}$ or $\mathrm{u} = \frac{\pi \mathrm{r}}{\mathrm{t}}$ ………………….. (1)

Let v and v' be the velocities of A and B respectively after impact.

Then, by the principle of momentum,

$\mathrm{mv} + \mathrm{mv}' = \mathrm{mu}$ (m being the mass of each ball)

i.e. $v + v' = u$ …………….. (2)

Also by the Newton's law, $\mathrm{v} - \mathrm{v}' = -\mathrm{e}(\mathrm{u} - 0)$

i.e. $\mathrm{v} - \mathrm{v}' = -\mathrm{eu}$ ……… (3)

Solving (2) and (3),

We get $\mathrm{v} = \frac{\mathrm{u}}{2}(1 - \mathrm{e})$; $\mathrm{v}' = \frac{\mathrm{u}}{2}(1 + \mathrm{e})$

Clearly v' is greater than v. hence B will move in advance of A. let it strike A again t_1 secs. After the first impact

The velocity of B relative to A,

After the first impact = $v' - v = eu$

From (3), before striking again, B should cover a distance equal length to the circumference relative to A.

Therefore $(v' - v).t_1 = 2\pi r$

i.e. $eu.t_1 = 2\pi r$

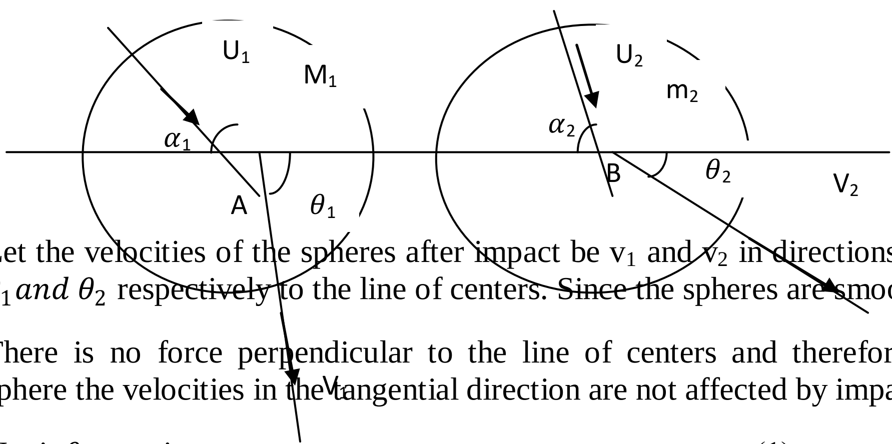
$$t_1 = \frac{2\pi r}{eu} = \frac{\frac{2\pi r}{e}}{e.\left(\frac{\pi r}{t}\right)} \quad \text{Using (1)}$$

$$= \frac{2t}{e}$$

12.5 OBLIQUE IMPACT OF TWO SMOOTH SPHERES

The second impact occurs $\frac{2t}{e}$ secs. After the first.

A smooth sphere of mass m_1 impinges obliquely with velocity u_1 on another smooth sphere of mass m_2 moving with velocity u_2. If the directions of motion before impact makes angle $\alpha_1 and\ \alpha_2$ respectively with the line joining the centers of the sphere and if the coefficient of restitution be e, to find the velocities and directions of motion after impact.



Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at $\theta_1 and\ \theta_2$ respectively to the line of centers. Since the spheres are smooth,

There is no force perpendicular to the line of centers and therefore, for each sphere the velocities in the tangential direction are not affected by impact

$V_1 \sin\theta_1 = u_1 \sin \alpha_1$ ………………………… (1)

$V_2 \sin \theta_2 = u_2 \sin \alpha_2$ ……………………… (2)

By Newton's law concerning velocities along the common normal AB,

$V_2 \cos \theta_2 = v_1 \cos \theta_1 = -e(u_2\cos\alpha_2 - u_1\cos\alpha_1)$ …………….. (3)

By the principle of conservation of momentum along AB

$m_1v_2\cos\theta_2 + m_1v_1\cos\theta_1 = m_2u_2\cos\alpha_2 + m_1u_1\cos\alpha_1$………. (4)

$(4) - (3) \times m_2$ gives

$$v_1\cos\theta_1.(m_1+m_2) = m_2u_2\cos\alpha_2 + m_1u_1\cos\alpha_1 + \ em_2(u_2\cos\alpha_2 - u_1\cos\alpha_1)$$

i.e. $v_1\cos\theta_1 = \frac{u_1\cos\alpha_1(m_1-em_2)+m_2u_2\cos\alpha_2(1+e)}{m_1+m_2}$ ……… (5)

$(4)+(3)\times m_1$ gives

$v_2\cos\theta_2 = \frac{u_2\cos\alpha_2(m_2-em_1)+m_1u_1\cos\alpha_1(1+e)}{m_1+m_2}$ ………… (6)

From (1) and (5) , by squaring and adding ,we obtain v_1^2 and by division ,we have tan θ_1.

Similarly from (2) and (6) we get v_2^2 and tan θ_2. Hence the motion after impact is completely determined.

Corollary: If the two spheres are perfectly elastic and of equal mass, then e = 1

And $m_1 = m_2$

Then from equations (5) and (6) we have

$$v_1\cos\theta_1 = \frac{0+m_1u_2\cos\alpha_2.2}{2m_1} = u_2\cos\alpha_2$$

And $v_2\cos\theta_2 = \frac{0+m_1u_1\cos\alpha_2.2}{2m_1} = u_1\cos\alpha_1$

Hence if two equal perfectly elastic spheres impinge, they interchange their velocities in the direction of the line of centers.

Corollary: 2

Usually, in most important problems on oblique impact, one of the spheres is at rest. Suppose m_2 is at rest i.e. $u_2 = 0$.

From equation (2)

$v_2 \sin\theta_2 = 0$. i.e. $\theta_2 = 0$.

Hence m_2 moves along AB after impact. This is seen independently ,since the only force on m_2 during impact is along the lines of centers

Corollary: 3

The impulse of the blow on the sphere A of mass m_1 = change of momentum of A along the common normal

$$= m_1(v_1\cos\theta_1 - u_1\cos\alpha_1)$$

$$= \ m_1\left[\frac{u_1\cos\alpha_1(m_1-em_2)+m_2u_2\cos\alpha_2(1+e)}{m_1+m_2} - u_1\cos\alpha_1\right]$$

$= m_1[\frac{m_1u_1\cos\alpha_1 - em_2u_1\cos\alpha_1 + m_2u_2\cos\alpha_2 + em_2u_2\cos\alpha_2 - m_1u_1\cos\alpha_1 - m_2u_1\cos\alpha_1}{m_1 + m_2}]$

$= \frac{m_1[m_2u_2\cos\alpha_2(1+e) - m_2u_1\cos\alpha_1(1+e)]}{m_1+m_2}$

$= \frac{m_1m_2(1+e)}{m_1+m_2}(u_2\cos\alpha_2 - u_1\cos\alpha_1)$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

12.6 LOSS OF KINETIC ENERGY DUE TO OBLIQUE IMPACT OF TWO SMOOTH SPHERES:

Two spheres of masses m_1 and m_2 moving along with velocities u_1 and u_2 at angles $\alpha_1 and\ \alpha_2$ with their collision. To find an expression for the loss of kinetic energy.

The velocities perpendicular to the line of centers are not altered by impact. Hence the loss of kinetic energy in the case of oblique impact is therefore the same as in the case of direct impact , the quantities by u_1 and u_2 and $u_1cos\alpha_1$ and $u_2\cos\alpha_2$ respectively .therefore the loss is ½ $(m_1m_2)/(m_1+m_2)$ $-(1-e^2)(u_1\cos\alpha_1 - u_2\cos\alpha_2)^2$

We shall now derive this independently.

Let v_1 and v_2 be the velocities of the spheres after impact, in directions inclined at angles $\theta_1$and $\theta_2$respectively to the line of centers. The tangential velocity of each sphere is not altered by impact.

$V_1 \sin\theta_1 = u_1 \sin \alpha_1$ ………………………… (1) And

$V_2 \sin \theta_2 = u_2 \sin \alpha_2$ ………………………………… (2)

By Newton's law

$V_2 \cos \theta_2 = v_1 \cos \theta_1 = -e(u_2cos\alpha_2 - u_1cos\alpha_1)$ …………………………….. (3)

By the principle of conservation of momentum

$$m_1v_2\cos\theta_2 + m_1v_1\cos\theta_1 = m_2u_2\cos\alpha_2 + m_1u_1\cos\alpha_1$$

i.e. $m_1(u_1\cos\alpha_1 - v_1\cos\theta_1) = m_2(v_2\cos\theta_2 - u2\cos\alpha2$ ……........................…. (4)

Change in K.E. $= \frac{1}{2} m_1{u^2}_1 + \frac{1}{2} m_2{u^2}_2 - \frac{1}{2} m_1{v^2}_1 - \frac{1}{2} m_2{v^2}_2$

$$= \frac{1}{2}\, m_1 {u_1}^2 \,(cos^2\alpha_1 + sin^2\alpha_1) + \frac{1}{2}\, m_2 {u_2}^2 (cos^2\alpha_2 + sin^2\alpha_2)$$

$$-\frac{1}{2}\, m_1 {v_1}^2 \,(cos^2\theta_1 + sin^2\theta_1) - \frac{1}{2}\, m_2 {v_2}^2 (cos^2\theta_2 + sin^2\theta_2)$$

$$= \frac{1}{2}\, m_1 {u_1}^2\, cos^2\alpha_1 + \frac{1}{2}\, m_2 {u_2}^2 cos^2\alpha_2$$

$$-\frac{1}{2}\, m_1 {v_1}^2\, cos^2\theta_1 - \frac{1}{2}\, m_2 {v_2}^2 cos^2\theta_2$$

Using (1) and (2)

$$= \frac{1}{2}\, m_1 ({u_1}^2 cos^2\alpha_1 - {v_1}^2\, cos^2\theta_1) + \frac{1}{2}\, m_2 ({u_2}^2 cos^2\alpha_2 - {v_2}^2 cos^2\theta_2)$$

$$= \frac{1}{2}\, m_1 (\mathrm{u_1 cos\alpha_1 + v_1 cos\theta_1})(\mathrm{u_1 cos\alpha_1 - v_1 cos\theta_1})$$

$$+\frac{1}{2}\, \mathrm{m_2(u_2 cos\alpha_2 + v_2 cos\theta_2)}(\mathrm{u_2} cos\alpha_2 - v_2 cos\theta_2)$$

$$= \frac{1}{2}\, m_1 (\mathrm{u_1 cos\alpha_1 + v_1 cos\theta_1})(u_1 cos\alpha_1 - v_1 cos\theta_1)$$

$$-\frac{1}{2}\, \mathrm{(u_2 cos\alpha_2 + v_2 cos\theta_2).\, m_1(u_1 cos\alpha_1 - v_1 cos\theta_1)}$$

Using (4)

$$= \frac{1}{2}\, \mathrm{m_1(u_1 cos\alpha_1 - v_1 cos\theta_1)(u_1 cos\alpha_1 + v_1 cos\theta_1}$$

$$\mathrm{- u_2 cos\alpha_2 - v_2 cos\theta_2)}$$

$$= \frac{1}{2}\, \mathrm{m_1(u_1 cos\alpha_1 - v_1 cos\theta_1)}$$

$$\mathrm{[u_1 cos\alpha_1 - u_2 cos\alpha_2 + e(u_2 cos\alpha_2 - u_1 cos\alpha_1)]}$$

using (3)

$$= \frac{1}{2}\, \mathrm{m_1(u_1 cos\alpha_1 - v_1 cos\theta_1)(u_1 cos\alpha_1 - u_2 cos\alpha_2)(1 - e)} \quad \ldots\ldots.. (5)$$

$$\frac{\mathrm{u_1 cos\alpha_1 - v_1 cos\theta_1}}{\mathrm{m_2}} = \frac{\mathrm{v_2 cos\theta_2 - u_2 cos\alpha_2}}{\mathrm{m_1}}$$

And each $= \frac{\mathrm{u_1 cos\,\alpha_1 - v_1 cos\,\theta_1 + v_2 cos\,\theta_2 - u_2 cos\,\alpha_2}}{\mathrm{m_1 + m_2}}$

$$= \frac{\mathrm{(u_1 cos\alpha_1 - u_2 cos\alpha_2) + (v_2 cos\theta_2 - v_1 cos\theta_1)}}{\mathrm{m_1 + m_2}}$$

$= \frac{\mathrm{u_1 cos\,\alpha_1 - u_2 cos\,\alpha_2 - e(u_2 cos\,\alpha_2 - u_1 cos\,\alpha_1)}}{\mathrm{m_1 + m_2}}$ Using (3)

$= \frac{(u_1\cos\alpha_1 - u_2\cos\alpha_2)(1+e)}{m_1+m_2}$

Therefore $u_1\cos\alpha_1 - v_1\cos\theta_1 = \frac{m_2(1+e)}{m_1+m_2}(u_1\cos\alpha_1 - u_2\cos\alpha_2)$

Substituting in (5)

Change in K.E. $= \frac{1}{2}\frac{m_2(1+e)}{m_1+m_2}(u_1\cos\alpha_1 - u_2\cos\alpha_2)$

$$\times (u_1\cos\alpha_1 - u_2\cos\alpha_2)(1-e)$$

$$= \frac{1}{2}\frac{m_2(1+e)}{m_1+m_2}(1-e^2)(u_1\cos\alpha_1 - u_2\cos\alpha_2)^2$$

If the spheres are imperfectly elastic, e = 1 and the loss of kinetic energy is Zero.

12.7 CHECK YOUR PROGRESS

1. Define the Principle of Conservation of Momentum?

2. What is meant by direct Impact?

3. What is meant by Oblique Impact?

4. Define line of Impact?

12.8 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The impulse of the force exerted by A on B is equal and opposite to that of the force exerted by B on A. it follows that the change in momentum of A is equal and opposite to the change in momentum of B, the moments being measured along the common normal. The sum of the moments of the bodies; measured along the common normal is altered by impact. This is called the principle of conservation of linear momentum.

2. When two bodies, moving along the same line, collide the impact is called direct impact.
3. When two bodies, moving along different lines, collide the impact is called oblique impact.
4. The instant when two bodies just collide, the line joining their centers is called the line of impact.

Notes

12.9 SUMMARY

- The total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

- The sum of the moments of the bodies; measured along the common normal is altered by impact. This is called the principle of conservation of linear momentum

- When two bodies, moving along the same line, collide the impact is called direct impact.

- When two bodies, moving along different lines, collide the impact is called oblique impact.

- The instant when two bodies just collide, the line joining their centers is called the line of impact.

12.10 KEYWORDS

Direct Impact – Oblique Impact- Loss of Kinetic Energy- Smooth Spheres

12.11 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. A sphere of mass m moving on a horizontal plane with velocity v impinges obliquely on a sphere of mass m^1 at rest on the same plane. If e = 1 , and $m = m^1$, prove that the directions of motion after impact are at right angles.
2. A smooth sphere impinges on another one at rest. After collision, their directions of motion are at right angles. Show that if they assumed perfectly elastic their masses must be equal.
3. If two equal perfectly elastic spheres impinge obliquely, prove that they interchange their velocities in the directions of the line of centers.
3. Two equal perfectly elastic balls impinge; if their directions of motion before impact be at right angles, show that their directions of motion after impact are also at right angles.
4. A sphere of mass m collides with a sphere of mass m_1 at rest, both spheres being smooth. After collision, their paths are at right angles. If e is the coefficient of restitution, prove that $m = em_1$.

12.12 FURTHER READINGS

Dr. M.K. Venkataraman, Statics, Agasthiar publications, 17th Edition, 2014.

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UNIT XIII SIMPLE HARMONIC MOTION

STRUCTURE

13.0 Introduction
13.1 Objectives
13.2 Velocity and Acceleration in Polar Coordinates
13.3 Equation of Motion
13.4 Note on Equiangular Spiral
13.5 Worked Example
13.6 Check your Progress
13.7 Answers to Check Your Progress Questions
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13.10 Self Assessment Questions and Exercises
13.11 Further Readings

13.0 INTRODUCTION

In the previous chapters, we have considered some particular cases of motion of a particle in two dimensions. To fix the position of a particle in a plane, we require two coordinates and to study the motion of the particle, we require its components velocities and accelerations in two mutually perpendicular directions. We had previously used Cartesian coordinates. In this chapter we shall use polar coordinates.

13.1 OBJECTIVES

After going through this unit, you will be able to:

- To learn about the meaning of central forces
- To learn about the path and accelerations of central forces
- To learn about the motion of a particle by using polar coordinates.

13.2 VELOCITY AND ACCELERATION IN POLAR COORDINATES

		Magnitude	Direction	Sense
1	Radial component of velocity	$\dot{r}$	Along the radius vector	In the direction in which r increases.
2	Transverse component velocity	$r\dot{\theta}$	Perpendicular to the Radius vector	In the direction in which the θ increases.
3	Radial component of acceleration	$\ddot{r} - r\dot{\theta}^2$	Along the radius vector.	In the direction in which the r increases.
4	Transverse component acceleration	$\frac{d}{dt}(r^2\dot{\theta})$	Perpendicular to the Radius vector	In the direction in which the θ increases.

Let P be the position of a moving particle at time t. taking O as the pole and OX as the initial line, let the polar coordinates of P be (r,θ). OP = r is the position vector of P. hence the velocity of P =$\frac{d}{dt}(r)$. Since r is the modulus and amplitudeθ,$\frac{d}{dt}(r)$. Will have components r along OP and r$\dot{\theta}$ to OP.

Hence the velocity vector v at P components $\dot{r}$ along OP in the direction in which r increases and r$\dot{\theta}$ perpendicular to OP in the direction in which θ increases. These are respectively called the radial and transverse components of v.

The acceleration vector at p is the derivative of the velocity vector v. the radial component of v is a vector with modulus r and amplitude θ. Hence the derivative of $\dot{r}$ will have components

$\frac{d}{dt}(\dot{r}) = \ddot{r}$ along OP in the direction in which r increases and

$\dot{r}\frac{d}{dt}\dot{(\theta)} = \dot{r}\dot{\theta}$ Perpendicular to OP I the direction in which θ increases.

The transverse component of v is vector with modulus r$\dot{\theta}$ and amplitude$\varphi = \frac{\pi}{2} + \theta$.

Hence the derivative of r$\dot{\theta}$ will have components

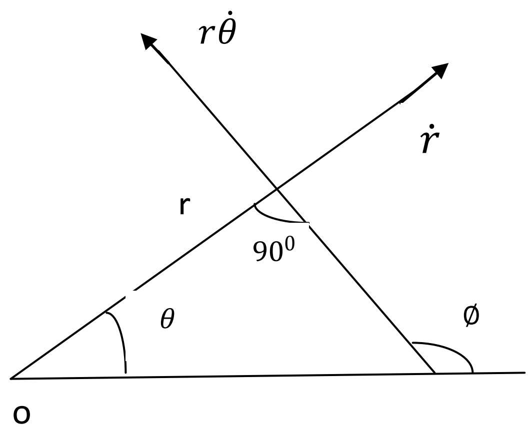
(i) $\frac{d}{dt}\left(\mathrm{r}\dot{\theta}\right) = r\ddot{\theta}$ + $\dot{\theta}\dot{r}$ along the line of r$\dot{\theta}$ i.e. in the direction perpendicular to OP and

(ii) $r\dot{\theta}\frac{d}{dt}\left(\frac{\pi}{2} + \theta\right) = r\dot{\theta}^2$ in the direction perpendicular to the line of r$\dot{\theta}$ in the direction PO.

Hence the totals of the components of acceleration are $\ddot{r} - r\dot{\theta}^2$ in the direction OP and $r\ddot{\theta}$+2$\dot{r}\dot{\theta}$ in the perpendicular direction.

Now $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = r\ddot{\theta}$+2$\dot{r}\dot{\theta}$

Therefore acceleration perpendicular to OP is also = $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$



Corollary :(1) suppose the particle P is describing a circle of radius ‘a’ . the r = a throughout the motion

Hence $\ddot{r}$ = 0 , and the radial acceleration = $\ddot{r} - r\dot{\theta}^2$

$= -a\dot{\theta}^2$

The acceleration perpendicular to OP $= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{a}\, a^2\,\ddot{\theta} \;= a\ddot{\theta}$

Hence for a particle describing a circle of radius a, the acceleration at any point P has the component at $\ddot{\theta}$ along the tangent at P and $\dot{\theta}^2$ along the radius to the centre.

(2) the magnitude of the resultant velocity of P,

$$\sqrt{\dot{r}^2 + \left(r\dot{\theta}\right)^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

And the magnitude of the resultant acceleration = $\sqrt{\left(\ddot{r} - r\dot{\theta}2\right)^2 + \left(\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\right)^2}$

13.3 EQUATION OF MOTION IN POLAR COORDINATES

If R and S are the components of the external force acting on a particle of mass m in the radial and transverse directions, we have the equations

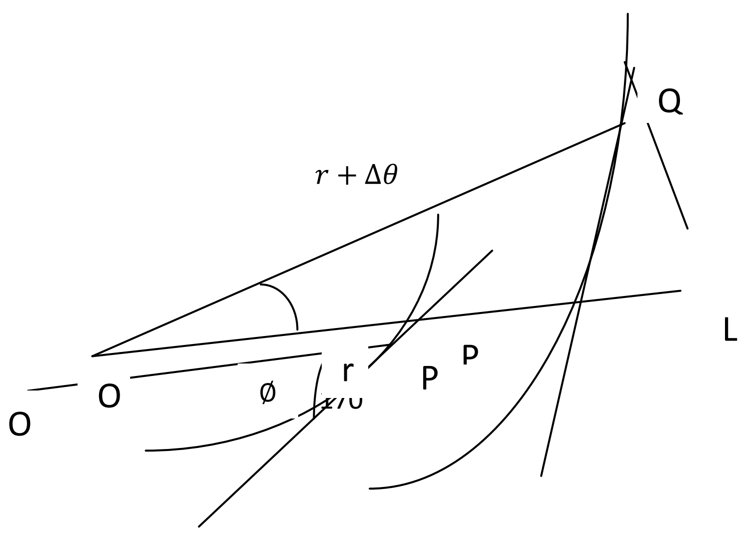
R = m $(\ddot{r} - r\dot{\theta}^2)$ ………………………………………… (1)

S = m. $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$…………………………………………
(2)

If R and S are known functions of the coordinate's r, θ and the time t, the differential equations (1) and (2) can be solved to find r and θ as functions of t and by eliminating t, the polar equation to the path is got.

13.4 NOTE ON EQUIANGULAR SPIRAL

Some questions in this chapter will relate to the curve called the equiangular spiral. This curve has the important property that the tangent at any point P on it makes a constant angle with the radius vector OP.



Let OP (=r) and OQ (= r+Δr) be two consecutive radii vectors such that the inclined angle POQ = $\Delta\theta$

Draw QL perpendicular to OP.

Then OL = (r+Δr) cos $\Delta\theta$ = r + Δr approximately .

Hence PL = OL$-$ OP = Δr

And LQ = (r+ Δr) sin $\Delta\theta$ = (r+Δr) $\Delta\theta$ nearly

= r $\Delta\,\theta$ to the first order of smallness

Hence tan ∠OPL = $\frac{QL}{PL} = r\frac{\Delta\theta}{\Delta r}$

In the limit as Δr and $\Delta\theta$ both tends to 0, the point Q tends to coincide with P. let φ be the angle made by the tangent at P with OP.

Then φ = $\lim_{Q\to P} \angle QPL$

Hence tan φ = $\lim_{\Delta r\to 0} tan\,\angle QPL = \lim\limits_{\Delta r\to 0} \mathrm{r}\,\frac{\Delta\theta}{\Delta r}$ = r $\frac{d\theta}{dr}$

i.e. tan φ =r $\frac{d\theta}{dr}$

This formula is an important one in dealing with curves in polar coordinates and it gives the angle between the radius vector and the tangent. Now for the equiangular spiral, at any point P on it the angle φ is constant.

Let $\varphi = \alpha$ then tan φ = tanα.

i.e. r $\frac{d\theta}{dr}$= tan α or $\frac{dr}{r} = cot\alpha.\,d\theta$

Integrating log r = $\theta cot\alpha + constant$

i.e. r = $a\ e^{\theta cot\alpha}$

this is the polar equation to the equiangular spiral.

13.5 WORKED EXAMPLE

Example: 1 the velocities of a particle along and perpendicular to a radius vector form a fixed origin are λr^2 and $\mu\theta^2$ where μ and λ are constants. Show that the equation to the path of the particle is $\frac{\lambda}{\theta} + c = \frac{\mu}{2r^2}$ where c is a constant.

Show that the accelerations along and perpendicular to the radius vector are

$2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ And $\mu(\lambda r\theta^2 + \frac{2\mu\theta^3}{r})$

Radial velocity = $\frac{dr}{dt} = \lambda r^2$

Transverse velocity = r $\frac{d\theta}{dt} = \mu\theta^2$

Divided (2) by (1) , we have

$$r\frac{d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2}$$

i.e. $\lambda\frac{d\theta}{\theta^2} = \frac{\mu}{r^3}dr$

Integrating $-\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} + C$

i.e. $\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + C$

(3) Is the equation to the path

Differentiating (1), $\frac{d^2r}{dt^2} = \lambda.2r\frac{dr}{dt} = 2\lambda^2 r^3$ using (1)

Radial acceleration = $\ddot{r} - r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$2\lambda^2 r^3 - r\left(\frac{\mu\theta^2}{r}\right)^2 = 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r} \text{ Using (2)}$$

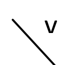
Transverse acceleration = $\frac{1}{r}\frac{d}{dt}(r^2\theta) = \frac{1}{r}\frac{d}{dt}(r^2\frac{\mu\theta^2}{r})$

$$= \frac{1}{r}\frac{d}{dt}(\mu r\theta^2) = \frac{\mu}{r}\left[r^2\theta\frac{d\theta}{dt} + \theta^2\frac{dr}{dt}\right]$$

$$= \frac{\mu}{r}\left[2r.\theta\frac{\mu\theta^2}{r} + \theta^2.\lambda r^2\right] = \mu\left[\frac{2\mu\theta^3}{r} + \lambda r\theta^2\right].$$

Example :2

Show that the path of a point P which possesses two constant velocity u and v first of which is in a fixed direction and the second of which is



perpendicular to the radius OP drawn from a fixed point O, is a conic whose eccentricity is $\frac{u}{v}$.

Take O as the pole and the line OXC parallel to the given direction as then initial line . P has two velocities u parallel to OX and v perpendicular to OP

Resolving the velocities along and perpendicular to OP ,we have

$\frac{dr}{dt} = u\ cos\theta$ ……………………………………(1)

$r\frac{d\theta}{dt} = v - u\sin\theta$…………………………………(2)

to get the equation to the path ,we have to eliminate t.

Dividing (2) by (1) , we have

$r\frac{d\theta}{dt} = \frac{v - u\sin\theta}{u\cos\theta}$

i.e. $\frac{u\ cos\ \theta}{v - usin\theta}\ d\theta = \frac{dr}{r}\ or \frac{d(usin\theta\ )}{v - usin\theta} = \frac{dr}{r}$

Integrating

$-\log(v - u\ sin\theta) + \log A = \log r$,where log A is the constant of integration .

i.e. log r + log (v – u sin θ)= log A

r(v – u sinθ)= A or $\frac{A}{r} = v - usin\ \theta$

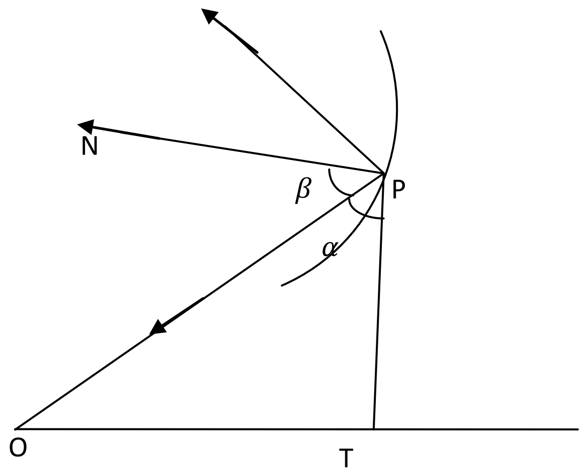
This is the form $\frac{l}{r} = 1 + ecos(\theta + \alpha)$

Comparing (1) and (2) we have $l = \frac{A}{v}$ $, e = \frac{u}{v}$ $and\ \alpha = 90^0$

We know from analytical geometry that (2) is the polar equation to conic whose focus is at the pole, semi- latus rectum is l, eccentricity is e and whose major axis makes an angle α with the initial line. Hence (1) is a conic whose focus is at O, semi-latus rectum is $\frac{A}{v}$, eccentricity is $\frac{u}{v}$ and whose major axis is perpendicular to the initial line.

Example 3:

A point P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as the distance from O; show that the curve is an equiangular spiral whose pole is O, and that the acceleration of the point is along the normal at P and varies inversely as OP.



Taking O as the pole ,let p be (r , θ),

Resultant velocity of P = $\sqrt{(\dot{r})^2 + r^2(\dot{\theta})^2}$ = constant (given)

If this velocity = k , we have

$$(\dot{r})^2 + r^2(\dot{\theta})^2 = k^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also angular velocity about O $\dot{\theta} = \frac{\lambda}{r}\,(given)$…………………………….(2)

Form (1) and (2),

$(\dot{r})^2 + r^2\frac{\lambda^2}{r^2} = k^2$

i.e. $(\dot{r})^2 = k^2 - \lambda^2$ or $\dot{r} = \sqrt{k^2 - \lambda^2}$……………………………….(3)

eliminate t from (2) and (3)

$\frac{dr}{d\theta} = \frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{\dot{r}}{\dot{\theta}} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}\,r$

i.e. $\frac{dr}{r} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}d\theta$

integrating log r = $\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta + B$ or $e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta+B} = e^B e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta}$

putting e^B = a and $\frac{\sqrt{k^2-\lambda^2}}{\lambda}$ = cot α ,

the above becomes $r = ae^{\theta cot\alpha}$ ……………………………….(4)

hence the path is an equiangular spiral , whose pole is O.

differentiating (3) , $\ddot{r} = 0$.

Radial acceleration = $\ddot{r} - r\dot{\theta}^2 \;\; = -r^2 \frac{\lambda^2}{(\dot{r})^2} = -\frac{\lambda^2}{r}$

The negative sign shows that the radial acceleration at P is along PO

Transverse acceleration = $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{\lambda}{r}\right) = \frac{1}{r}.\frac{d}{dt}\,(r\lambda)$

$$= \frac{\lambda}{r}\,\dot{r} = \frac{\lambda}{r}\sqrt{k^2-\lambda^2} \text{ From (3)}$$

Resultant acceleration of P = $\sqrt{\left(-\frac{\lambda^2}{r}\right)^2 + \left[\frac{\lambda}{r}\sqrt{k^2-\lambda^2}\right]^2}$

$$= \sqrt{\frac{\lambda^4}{r^2} + \frac{\lambda^2}{r^2}(k^2-\lambda^2)} = \sqrt{\frac{\lambda^2 k^2}{r^2}} = \frac{\lambda k}{r}$$

Thus the resultant acceleration varies inversely as r i.e. as OP. let this acceleration be along PN making an angle with PO.

$$tan\beta = \frac{compone\ nt\ perpendicular\ \ to\ PO}{component\ \ along\ \ PO}$$

$= \frac{\frac{\lambda}{r}\sqrt{k^2-\lambda^2}}{\frac{\lambda^2}{r}} = \frac{\sqrt{k^2-\lambda^2}}{\lambda}$

But $\cot\alpha = \frac{\sqrt{k^2-\lambda^2}}{\lambda}$ from equation (4)

Hence $\tan\beta = \cot\alpha = \tan(90^0 - \alpha)$

i.e. $\beta = 90^0 - \alpha$ or $\beta + \alpha = 90^0$

Hence angle NPT = 90^0 where PT is the tangent at P.

Hence PN is the normal at P.

13.6 CHECK YOUR PROGRESS

1. What is meant by Radial Transverse?
2. Write the equation of polar coordinates?
3. Define Center of Forces?

13.7 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The velocity vector v at P components $\dot{r}$ along OP in the direction in which r increases and r$\dot{\theta}$ perpendicular to OP in the direction in which θ increases. These are respectively called the radial and transverse components of v.

2. The Equation of Polar Coordinate is given by

R = m ($\ddot{r} - r\dot{\theta}^2$) …………………………………………… (1)

S = m. $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$……………………………………………… (2)

If R and S are known functions of the coordinate's r, θ and the time t

4. A particle describes a path, acted on by an attractive force F towards a fixed point O. such a force is called central force and the path described by the particle is called a central orbit. The fixed point is known as the centre of the force

13.8 SUMMARY

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- To fix the position of a particle in a plane, we require two coordinates and to study the motion of the particle, we require its components velocities and accelerations in two mutually perpendicular directions.
- The magnitude of the resultant velocity of P,

$$\sqrt{\dot{r}^2 + \left(r\dot{\theta}\right)^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

- The velocity vector v at P components $\dot{r}$ along OP in the direction in which r increases and r$\dot{\theta}$ perpendicular to OP in the direction in which θ increases. These are respectively called the radial and transverse components of v.
- A particle describes a path, acted on by an attractive force F towards a fixed point O. such a force is called central force and the path described by the particle is called a central orbit. The fixed point is known as the centre of the force
- tan φ =r $\frac{d\theta}{dr}$, This formula is an important one in dealing with curves in polar coordinates and it gives the angle between the radius vector and the tangent.

13.9 KEYWORDS

Velocity and Acceleration- Polar Coordinates- Equation of Motion- Simple Harmonic Motion.

13.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. The velocities of a particle along and perpendicular to the radius vector from a fixed origin are a and b. Find the path and the accelerations along and perpendicular to the radius vector.

2. If a point moves so that its radial velocity is k times its transverse velocity, show that its path is an equiangular spiral.

3. If the angular velocity of a particle about a point in its plane of motion be constant, prove that the transverse component of its acceleration is proportional to the radial component of its velocity.

4. A point moves in a parabola in such a manner that the component velocity at right angles to the Radius vector form the focus is constant. Show that the acceleration of the point is constant in magnitude.

5. A point P describes an equiangular spiral with constant angular velocity about the pole O; show that its accelerations varies as OP and is in a direction making with the tangent at P the same constant angle that OP makes.

13.11 FURTHER READINGS

Dr. M.K. Venkataraman, Statics, Agasthiar publications, 17^{th} Edition, 2014.

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UNIT XIV CENTRAL ORBITS

STRUCTURE

14.1 Introduction
14.1 Objectives
14.2 Differential Equation of Central Orbits
14.3 Perpendicular Pole from the Tangent
11.4 Pedal equation of Central Orbit
14.5 Pedal Equation of Well Known Curves
14.6 Velocities in Central Orbits
14.7 Two Folded Problem
14.8 Check your Progress
14.9 Answers to Check Your Progress Questions
14.10 Summary
14.11 Keywords
14.12 Self Assessment Questions and Exercises
14.13 Further Readings

14.0 INTRODUCTION

A particle describes a path, acted on by an attractive force F towards a fixed point O. such a force is called central force and the path described by the particle is called a central orbit. The fixed point is known as the centre of the force. Usually the magnitude of the central attraction F is a function only of the distance r of the particle from O. In such a motion, the particle must be always moving only in the plane containing O and the tangent at any point on its path, since there is no component of attraction perpendicular to the above plane. Hence central orbit must be a plane curve.

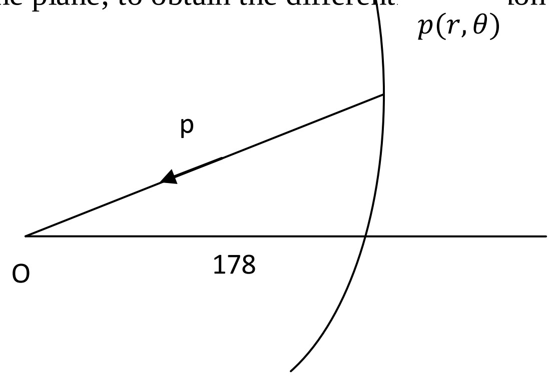
14.1 OBJECTIVES

After going through this unit, you will be able to:

- To learn about the Differential Equation of Central Orbits
- To learn about the Pedal equations of Orbits
- To learn about the velocity and well known curves of Pedal equations

14.2 DIFFERENTIAL EQUATIONS OF CENTRAL ORBITS

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane; to obtain the differential equation of its path.



Take O as the pole and a fixed line through O as the initial line. Let P(r,θ) be the polar coordinates of the particle at time t and m be its mass. Also let P be the magnitude of the central acceleration along PO.

The equations of motion of the particle are

$$m(\ddot{r} - r\dot{\theta}^2) = -mP \quad \text{i.e.} \quad (\ddot{r} - r\dot{\theta}^2) = -P \quad \text{……………………. (1)}$$

$$\text{And } \frac{m}{r} . \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \ \text{ i.e.} \frac{1}{r} . \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \quad \text{…………………... (2)}$$

Equation (2) follows from the fact that as there is no force at right angles to OP , the transverse component of the acceleration is zero throughout the motion.

From (2) , $r^2\dot{\theta}$ = constant = h (say)

To get the polar equation of the path, we have to eliminate the element of time between equations (1) and (3). For this purposes, it is found convenient to put $u = \frac{1}{r}$ and work with u as the dependent variable.

From (3) , $\dot{\theta} = \frac{h}{r^2} = hu^2$

Also $\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta} . \frac{d\theta}{dt}$

$$= -\frac{1}{u^2}\frac{du}{d\theta} . hu^2 = -h\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right).\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{dt^2} . hu^2 = -\frac{h^2u^2d^2u}{d\theta^2}$$

Substitute r and $\dot{\theta}$ in (1) , we get

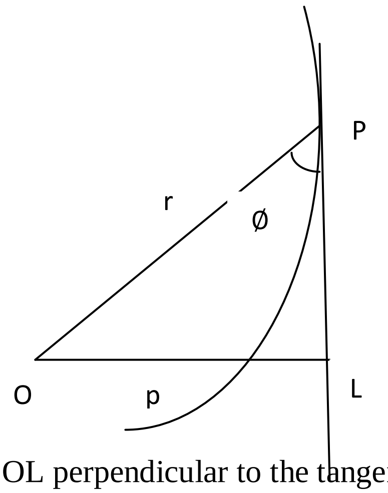
$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u} h^2u^4 = -p \text{ or } h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = p \quad \text{or } u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2}$$

This is the differential equation of a central orbit, in polar coordinates.

Note: if the central force is a repulsive one in a particular problem, the sign of P in (4) must be changed.

14.3 PERPENDICULAR POLE FROM THE TANGENT

Let φ be the angle made by tangent at P with the radius vector OP. we know that $\tan\varphi = r\frac{d\theta}{dr}$ ……………………………(1)



From O draw OL perpendicular to the tangent at P and let OL = P.

Then sin $\varphi = \frac{OL}{OP} = \frac{p}{r}$ or p = r sin φ ………………………….. (2)

Let us eliminate φ between (1) and (2) .

From (2) , $\frac{1}{p^2} = \frac{1}{r^2 sin^2\varphi} = \frac{1}{r^2}\, cossec^2\varphi$

$$= \frac{1}{r^2}\,(1 + cot^2\varphi)$$

$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right], \text{ substituting from (1)}$$

i.e. $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ …………………………….. (3)

Using $r = \frac{1}{u}, \ \frac{dr}{d\theta} = \frac{dr}{du} \cdot \frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

Hence (3) becomes

$$\frac{1}{p^2} = u^2 + u^4 . \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2 \quad \text{i.e. } \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

14.4 PEDAL EQUATION OF CENTRAL ORBIT

In certain curves the relation between P (the perpendicular from the pole on the tangent) and r (radius vector) is very simple. Such a relation is called the pedal equation or the (p , r) equation to the curve. We can get the (p , r) equation to a central orbit as follows:

In the usual notation, we have

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Differentiating both sides of (1) with respect to,

$$-\frac{2}{p^3}\cdot\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\ (u + \frac{d^2u}{d\theta^2}) \quad \ldots\ldots\ldots\ldots\text{.. (2)}$$

But the differential equation in polars is $u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2}$

Hence (2) becomes $-\frac{1}{p^3}\cdot\frac{dp}{d\theta} = \frac{p}{h^2u^2}\frac{du}{d\theta}$

i.e. $-\frac{1}{p^3}\ dp = \frac{P}{h^2u^2}du = \frac{P}{h^2}\ r^2\ d\left(\frac{1}{r}\right)$

$$= \frac{Pr^2}{h^2} \times -\frac{1}{r^2}\ dr = -\frac{P}{h^2}\ dr$$

Or $\frac{h^2}{p^3}\cdot\frac{dp}{dr} = P.$ ……………………………….. (3)

(3) is the (p , r) equation or the pedal equation to the central orbit

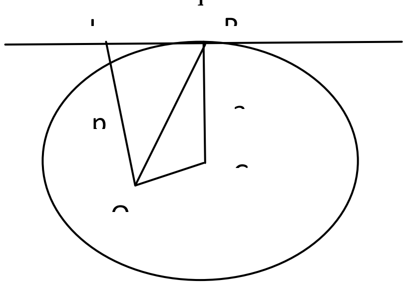
14.5 PEDAL EQUATION OF THE SOME OF THE WELL KNOWN CURVES

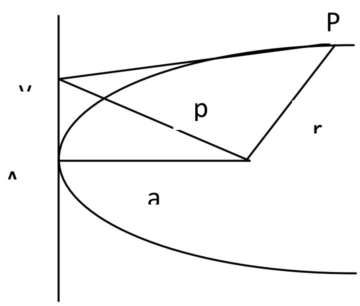
(1) Circle – pole at any point:

Let C be the center , a be the radius , O the pole where OC = c.

Let P be the any point on the circle and OL be the perpendicular from O on the tangent at P.

OP = r and OL = p .





From ΔOPC , $c^2 = r^2 + a^2 - 2ra\cos\angle OPC = r^2 + a^2 - 2ra\cos\angle POL$

Notes

$$= r^2 + a^2 - 2ra.\frac{p}{r} = r^2 + a^2 - 2ap$$

Hence the pedal equations of the circle for general position of the pole are $c^2 = r^2 + a^2 - 2ap$. When c = a, the pole is on the circumference and the equation is $r^2 = 2ap$.

(2) Parabola- pole at focus:

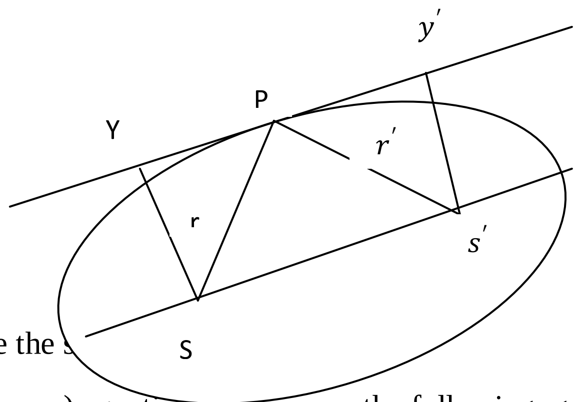
To get the (p , r) equation to a parabola , we assume the geometrical property that if the tangent at P meets the tangent at the vertex A in Y and S is the focus , then SY is perpendicular to PY and the triangle SAY are similar.

Hence $\frac{SA}{SY} = \frac{SY}{SP}$

i.e. $\frac{a}{p} = \frac{p}{r}$ or $p^2 = ar$

(3) Ellipse or Hyperbola – pole at focus:

Let S and S' be the foci of the ellipse and SY , S', Y' be the perpendiculars to the tangent at P. taking S as the pole , let SP = r, $S'P = r'$, $SY = p,\ S'Y' = p'$.



Let a and b be the

To find the (p , r) equation , we assume the following geometrical properties of the ellipse .

(i) $SP = S'P = 2a$ i.e. $r + r' = 2a$

(ii) $SY.S'Y' = b^2$ i.e. $pp' = b^2$

(iii) The tangent at P is equally inclined to the focal distances so that SPY and $S'\ P\ Y'$ are similar triangles.

So we have $\frac{p}{r} = \frac{p'}{r'}$

Now $\frac{b^2}{p^2} = \frac{pp'}{p^2}$ using (ii)

$= \frac{p'}{p} = \frac{r'}{r}$ using (iii)

$= \frac{2a-r}{r}$ using (1)

$= \frac{2a}{r} - 1$

Hence $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is the (p, r) equation to the ellipse.

By a similar argument, the (p, r) equation of the branch of the hyperbola nearer to the focus $\frac{b^2}{p^2} = \frac{2a}{r} + 1$

(4) Equiangular spiral:

In $p = r\, sin\varphi$ in the usual notation.

In the equilibrium spiral, $\varphi =$ constant = $\alpha(say)$

Hence $p = sin\alpha = kr$ is the (p, r) equation to the spiral

14.6 VELOCITIES IN A CENTRAL ORBIT

In every central orbit the areal velocity is constant and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Let at time t the particle be at P (r , θ) and at time t+ Δt, let it be at Q (r + $\Delta r, \theta + \Delta\theta$) .

sectional area OPQ described by the radius vector OP = Area of ΔOPQ nearly

$$= \frac{1}{2}\, OP.OQsin\angle POQ$$

$$= \frac{1}{2}\, r(r + \Delta r)\sin\Delta\theta$$

$= \frac{1}{2}\, r^2\, \Delta\theta$, to the first o In every central orbit the areal velocity is constant and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Let at time t the particle be at P (r , θ) and at time t+ Δt, let it be at Q (r + $\Delta r, \theta + \Delta\theta$) .

sectional area OPQ described by the radius vector OP = Area of ΔOPQ nearly

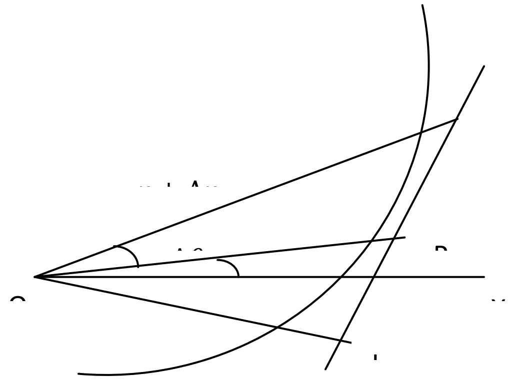
$= \frac{1}{2}\, OP.OQsin\angle POQ$

Notes

$=\frac{1}{2}\ r(r+\Delta r)\sin\Delta\theta$

$=\frac{1}{2}\ r^2\ \Delta\theta$, to the first order of smallness.

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle .



Hence in the central orbit, areal velocity of P

$= \lim_{\Delta t\to 0}\frac{1}{2}\ r^2\frac{\Delta\theta}{\Delta t}=\frac{1}{2}\ r^2\frac{d\theta}{dt}=\frac{1}{2}\ h$ ………………………….. (1)

Since $r^2\dot{\theta}=$ constant = h from equation (3)

Hence h = twice the areal velocity and as h is a constant, the areal velocity is constant. In other words, equal areas are described by the radius vector in equal times.

We can get another expression for the areal velocity.

Let Δs be the length of the arc PQ. Draw OL perpendicular to PQ. Sectional area POQ = ΔPOQ nearly = $\frac{1}{2}\ PQ.OL$

As tends to 0, Q tends to coincide with P along the curve and the chord QP becomes the tangent at P. length PQ = Δs , nearly and OL becomes the perpendicular from O on the tangent at P. let OL= P

Hence the areal velocity = $\lim_{\Delta t\to 0}\frac{1}{2}\frac{\Delta s}{\Delta t}\ .P=\frac{1}{2}\ P\frac{ds}{dt}=\frac{1}{2}\ pv$ …….. (2)

As $\frac{ds}{dt}$ is the rate pf describing s and so as the linear velocity of P.

Hence combining (1) and (2) , areal velocity = $\frac{1}{2}\ h=\frac{1}{2}\ pv.$ or $h=pv$ (i.e.) v = $\frac{h}{p}$.

Hence linear velocity varies inversely as OP.

14.7 TWO FOLD PROBLEMS CENTRAL ORBITS

It is clear that two types of problems arise in connection with central orbits. They are

(i) Given the orbit, to find the law of force to the pole.

(ii) Given the law of force, to find the path.

We shall first take up (i)

The differential equation to the central orbit in polar coordinates is

$$u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2}$$

Hence $p = h^2u^2(u + \frac{d^2u}{d\theta^2})$

Since the orbit is given, u is known as a function ofθ. Hence by differentiation, P can be got from the above equation.

In a few cases we may know the (p , r) equation to the path. To find P, we can use the equation

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}$$

Example 2:

Find the law of force towards the pole under which the curve $r^n = a^n cosn\theta$ can be described.

$r^n = a^n cosn\theta$

Since $r = \frac{1}{u}$, the equations $u^n a^n \cos n\,\theta = 1$ ………………………. (1)

Taking algorithms,

$$n \log u + n \log a + \log \cos n\,\theta = 0$$

Differentiating (2) with respect toθ.

$n.\frac{1}{u}\frac{du}{d\theta} - n\frac{sinn\theta}{cosn\theta} = 0$ i.e. $\frac{du}{d\theta} = u\,tan\,n\,\theta$ ………………… (3)

Differentiating (2) with respect to θ,

$$\frac{d^2u}{d\theta^2} = u\,n\,sec^2 n\,\theta + \tan n\,\theta \,.\frac{du}{d\theta}$$

$$= nu\,sec^2 n\,\theta + u\,tan^2 n\,\theta \text{ Using (2)}$$

$$u+\frac{d^2u}{d\theta^2}=u+nu\,sec^2n\,\theta+u\,tan^2n\,\theta$$

$$=nu\,sec^2n\,\theta+u(1+tan^2n\,\theta)$$

$$=\ nu\,sec^2n\,\theta+u\,sec^2n\,\theta$$

$$=(n+1)u\,sec^2n\,\theta$$

$$=(n+1)u.\,a^{2n}u^{2n+1}$$

$$P=\ h^2u^2\ \left(u+\frac{d^2u}{d\theta^2}\right)=\ h^2u^2.(n+1)u.\,a^{2n}u^{2n+1}$$

$$=(n+1)\ a^{2n}u^{2n+3}\ h^2$$

$$=\ (n+1)\ a^{2n}h^2.\frac{1}{\Gamma^{2n+3}}\ \text{.............................. (3)}$$

i.e. p is directly proportional to $\frac{1}{\Gamma^{2n+3}}$ which means the central acceleration varies inversely as the (2n+3)rd power of the distance.

Note:

From (3), p is positive only when n+1 > 0

i.e. n > -1.

For values of n < -1 , P will be negative and in such cases the central forces will be a repulsive one. The above case is a comprehensive one, giving the law of force for describing the follow well-known curves corresponding to particular values of n.

(i) When n = 1 , the equation is $r=a\cos\theta$. The curve is a circle and P $\alpha\ \frac{1}{r^5}$

(ii) When n = 2 , the equation is $r^2=a^2\cos 2\,\theta$. This is the lemniscates of Bernoulli and P $\alpha\,\frac{1}{r^7}$

(iii) When n = $\frac{1}{2}$, the equation is $r^{\frac{1}{2}}=a^{\frac{1}{2}}\cos\frac{\theta}{2}$, this is a cardioids and P $\alpha\ \frac{1}{r^4}$

(iv) When n =$-\ \frac{1}{2}$, the equation is $r^{-\frac{1}{2}}=a^{-\frac{1}{2}}\cos\frac{\theta}{2}$ i.e. $a^{\frac{1}{2}}=r^{\frac{1}{2}}\cos\frac{\theta}{2}$

So $\frac{2a}{r}=1+\cos\theta$, this is a parabola and P $\alpha\,\frac{1}{r^5}$

(v) When $n=\ -2$, the equation is $r^{-2}=a^{-2}\cos 2\,\theta$ i.e. $r^2\cos 2\,\theta=a^2$.

This is a rectangular hyperbola. In this case the actual value of $P=\ -a^{-4}h^2r$. The negative sign of P shows that the central force is a repulsive one.

Example 2:

Find the law of force to an internal point under which a body will describe a circle.

The pedal equation of the circle for a general position of the pole is

$$c^2 = r^2 + a^2 - 2ap \text{.................................. (1)}$$

Differentiating with respect to r,

$0 = 2r - 2a\frac{dp}{dr}$ i.e. $\frac{dp}{dr} = \frac{r}{a}$

Now the central acceleration

$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2 r}{p^3 a} = \frac{8h^2a^2r}{(r^2+a^2-c^2)^3}$ Substituting for p from (1)

Example 3:

A particle moves in a curve under a central attraction so that its velocity at any point is equal to that in a circle at the same distance and under the same distance and under the same attraction . show that the path is an equiangular spiral and that the law of force is that of the universe cube.

Let the central acceleration be P. if v is the velocity in a circle at a distance r under the normal acceleration P, then

$$\frac{v^2}{p} = P \text{ i.e. } v^2 = Pr \text{ .... (1)}$$

Since v is also the velocity in the central orbit,

$h = pv$ or $v = \frac{h}{p}$

Putting this is (1) , $\frac{h^2}{p^2} = Pr$ ....(2)

We know that $P = \frac{h^2}{p^3} . \frac{dp}{dr}$ .... (3)

Substituting (3) in (2) ,

$\frac{h^2}{p^2} = \frac{h^2}{p^3} . \frac{dp}{dr} . r$ i.e. $\frac{dp}{p} = \frac{dr}{r}$

Integrating, log p = log r + log A i.e. p = A r .... (4)

(4) Clearly the (p , r) equation to an equiangular spiral.

From (4), $\frac{dp}{dr} = A$. Substituting this in (3),

Notes

$P = \frac{h^2}{p^3} . A = \frac{Ah^2}{A^3 p^3}$ Using (4)

$= \frac{h^2}{A^2}\left(\frac{1}{r^3}\right)$ i.e. $P \alpha 1/r^3$.

The following three general principles hold good when two smooth moving bodies make an impact.

14.8 CHECK YOUR PROGRESS

1. Define central Orbit?
2. What is Velocity in central Orbit?
3. Write the Pedal Equation of central orbit?
4. Give the Differential Equation of Central Orbits?

14.9 ANSWERS TO CHECK YOUR PROGRESS

1. A particle describes a path, acted on by an attractive force F towards a fixed point O. such a force is called central force and the path described by the particle is called a central orbit
2. In every central orbit the areal velocity is constant and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path
3.The Pedal equation of Central Orbit given as $\frac{h^2}{p^3} . \frac{dp}{dr} = P$
4 The Differential Equation of Central Orbits is given by

$$u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2}$$

14.10 SUMMARY

- A particle describes a path, acted on by an attractive force F towards a fixed point O. such a force is called central force and the path described by the particle is called a central orbit. The fixed point is known as the centre of the force

- In every central orbit the areal velocity is constant and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path

- The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle

- The Pedal equation of Central Orbit given as $\frac{h^2}{p^3} . \frac{dp}{dr} = P$
- The Differential Equation of Central Orbits is given by

$$u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2}$$

14.11 KEYWORDS

Central orbits – Pedal Equations – Tangent – Equation of Well Known Curves – velocity.

14.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Find the law of forces when a circle id described under a central force to a point O on its circumference.
2. The velocity at any point of a central orbit is one half of what it would be for a circular orbit at the same distance. Obtain the law of forces
3. Using pedal equations, find the law of force towards the pole under which the following curves described: (i) Parabola (ii) Ellipse (iii) Hyperbola.
4. If the radius vector to a point which is describing an orbit sweeps out equal areas in equal times shows that the acceleration acting on the particle must be directed along the radius vector.
5. A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

14.13 FURTHER READINGS

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