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FUNCTIONAL ANALYSIS

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SYLLABI-BOOK MAPPING TABLE

Functional Analysis

Syllabi	Mapping in Book
UNIT 1 : VECTOR SPACE Vector Space; Subspace; Basis; Normed space; Basis; Normed Space	**Unit-1:** Vector Space **(Pages 1-15)**
UNIT 2: CONVEX SET Convex Sets; Norm Topology; Coset, Quotient Space; Equivalent Norms	**Unit-2:** Convex Set **(Pages 16-22)**
UNIT 3: NORMED SPACES AND SUBSPACES Finite Dimensional Normed Spaces and Subspaces; Compactness	**Unit-3:** Normed Spaces and Subspaces **(Pages 23-31)**
UNIT 4: LINEAR OPERATOR Linear Operator; Bounded Linear Operator	**Unit-4:** Linear Operator **(Pages 32-40)**
UNIT 5: LINEAR FUNCTIONAL Linear Functionals; Normed Spaces of Operators	**Unit-5:** Linear Functional **(Pages 41-50)**
UNIT 6: BOUNDED OR CONTINUOUS LINEAR OPERATOR Bounded or Continuous Linear Operator; Dual Space	**Unit-6:** Bounded or Continuous Linear Operator **(Pages 51-59)**
UNIT 7: INNER PRODUCT SPACE Inner Product; Orthogonal; Orthogonal Set; Orthonormal; Gram Schmidt's Process; Total Orthonormal Set	**Unit-7:** Inner Product Space **(Pages 60-74)**
UNIT 8: ANNIHILATORS AND PROJECTIONS Annihilator; Orthogonal Projection	**Unit-8:** Annihilators and Projections **(Pages 75-78)**
UNIT 9: HILBERT SPACE Hilbert Space; Convex Set; Direct Sum; Orthogonal; Isomorphic; Hilbert Dimension	**Unit-9:** Hilbert Space **(Pages 79-85)**
UNIT 10: REFLEXIVITY OF HILBERT SPACES Reflexivity of Hilbert Spaces	**Unit-10:** Reflexivity of Hilbert Spaces **(Pages 86-92)**
UNIT 11: RIESZ'S THEOREM Riesz's Theorem; Sesquilinear Form (or) Function; Riesz's Representation Theorem; Adjoint Operator; Properties of Adjoint Operators; Classification of Bounded Linear Operator	**Unit-11:** Riesz's Theorem **(Pages 93-107)**
UNIT 12: ADJOINT OPERATOR IN NORMED SPACES Hahn-Banach Theorem for Real Vector Space; Adjoint Operator in Normed Spaces; Relation between Adjoint Operator T* and Hilbert Adjoint Operator T*; Matrix; Baire's Category Theorem; Uniform Boundedness Theorem	**Unit-12:** Adjoint Operator in Normed Spaces **(Pages 108-120)**
UNIT 13: STRONG AND WEAK CONVERGENCE Strong and Weak Convergence; Convergence of Sequences of Operator and Functionals	**Unit-13:** Strong and Weak Convergence **(Pages 121-130)**
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UNIT I: VECTOR SPACE

Structure

1.1 INTRODUCTION

Functional Analysis is the study of vector spaces endowed with topological structures (that are compatible with the linear structure of the space) and of (linear) mappings between such spaces. Throughout this unit we will be working with vector spaces whose underlying field is the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. We begin our unit with some basic definitions, results and examples from linear algebra. Linear space also known as vector spaces arise naturally in applications, where in a physical problem is often modelled as a mathematical equation. The theory of Normed spaces, in particular Banach spaces, and the theory of linear operators defined on them are the most highly developed parts of functional analysis.

1.2 OBJECTIVES

Students will be able to

To understand Normed space.

Describe completion of a Normed space.

To differentiate Linearly independent and Linearly dependent.

To solve the results related to Normed space.

1.3 VECTOR SPACE

A vector space is a nonempty set V over a field F together with two operations + and **.** are defined called vector addition and scalar multiplication.

The operation + is defined by $+ : V \times V \to V$, (vector addition) which is satisfy the following conditions:

1. *Closure:* If **u** and **v** are any vectors in V, then the sum **u** + **v** belongs to V

2. *Commutative law:* For all vectors **u** and **v** in V, **u** + **v** = **v** + **u**

3. *Associative law:* For all vectors **u**, **v**, **w** in V, **u** + (**v** + **w**) = (**u** + **v**) + **w**

4. *Additive identity:* The set V contains an *additive identity* element, denoted by **0**, such that for any vector **v** in V, **0** + **v** = **v** and **v** + **0** = **v**.

5. *Additive inverses:* For each vector **v** in V, the equations **v** + **x** = **0** and **x** + **v** = **0** have a solution **x** in V, called an *additive inverse* of **v**, and denoted by **- v**.

The operation '.' is defined by $. : V \times V \to V$, (scalar multiplication) which is satisfy the following conditions:

1. *Closure:* If **v** in any vector in V, and c is any real number, then the product c · **v** belongs to V.

2. *Distributive law:* For all real numbers c and all vectors **u**, **v** in V, c · (**u** + **v**) = c · **u** + c · **v**

3. *Distributive law*: For all real numbers c, d and all vectors **v** in V, (c+d) · **v** = c · **v** + d · **v**

4. *Associative law*: For all real numbers c,d and all vectors **v** in V, c · (d · **v**) = (cd) · **v**

5. *Unitary law*: For all vectors **v** in V, 1 · **v** = **v**

Example:

(1) **Space $\mathbb{R}^n$:** This is the Eucliden space, the underlying set being the set of all n-tuples of real numbers, written $x = (\varepsilon_1, \dots, \varepsilon_n)$, $y = (\delta_1, \dots, \delta_n)$, etc., and we now see that this is a real vector space $(K = \mathbb{R})$ with two algebraic operations defined by

$$x + y = (\varepsilon_1 + \delta_1, \dots, \varepsilon_n + \delta_n)$$
$$\alpha x = (\alpha\varepsilon_1, \dots, \alpha\varepsilon_n), \quad (\alpha \in \mathbb{R})$$

1.4 SUBSPACE

A subspace of a vector space X is a nonempty subset Y of X such that for all $y_1, y_2 \in Y$ and all scalars α, β we have $\alpha y_1 + \beta y_2 \in Y$. Hence Y is itself a vector space, the two algebraic operations being those induced from X.

A special subspace of X is the improper subspace Y=X. Every other subspace of X $(\neq \{0\})$ is called proper. Another special subspace of any vector space X is Y ={0}.

1.5 LINEAR COMBINATION

A linear combination of vectors $x_1, \dots, x_m$ of a vector space X is an expression of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$, Where the coefficients $\alpha_1, \dots, \alpha_m$ are any scalars.

1.6 LINEAR SPAN

For any nonempty subset $M \subset X$ the set of all linear combinations of vector of M is called the span of M, written span M.

1.6.1 Linear Independence and Linear Dependence

Let M be the set of vectors $x_1, \dots, x_r$ $(r \geq 1)$ in a vector space X are defined by means of the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = 0, \dots\dots\dots(1)$$

Where $\alpha_1, \dots, \alpha_r$ are scalars. Clearly, equation (1) holds for $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0.$

If this is the only r-tuple of scalars for which (1) holds, the set M is said to be linearly independent. M is said to be linearly dependent if M is not linearly independent, that is, if (1) also holds for some r-tuple of scalars, not all zero.

1.6.2 Finite and Infinite Dimensional Vector Spaces

A vector space X is said to be finite dimensional if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of $n+1$ or more vectors of X is linearly dependent. n is called the dimension of X, written $n = \dim X.$ By definition X={0} is finite dimensional and dim X=0. If X is not finite dimensional, it is said to be infinite dimensional.

1.7 BASIS

If X is any vector space, not necessarily finite dimensional, and B is a linearly independent subset of X which spans X, then B is called a basis (or Hamel basis) for X.

Note:

Hence if B is a basis for X, then every nonzero $x \in X$ has a unique representation as a linear combination of elements of B with nonzero scalars as coefficients.

Theorem: 1.7.1 (Dimension of a subspace)

Let X be an n-dimensional vector space. Then any proper subspace Y of X has

dimension less than n.

Proof:

If n = 0, then X = {0} and has no proper subspace.

If dim Y = 0, then Y = {0}, and X $\neq$Y implies dim X $\geq$1.

Clearly, dim Y$\leq$ dim $X = n$. If dim Y were n, then

Y would have a basis of n elements, which would also be a basis for X since dim X = n, so that X = Y.

This shows that any linearly independent set of vectors in Y

must have fewer than n elements, and dim Y < n.

1.8 NORMED SPACE

Let X be a real or complex vector space. A real valued function ||.|| is said to be a norm on X if

$(i) \|x\| \geq 0, \quad \forall x \in X$

$(ii) \|x\| = 0 \quad iff\ x = 0$

$(iii) \|\alpha x\| = |\alpha| \|x\|, \ \forall \alpha \in K \ and \ \forall x \in X$

$(iv) \|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in X$

Where K is either $\mathbb{R}\ or\ \mathbb{C}$

Then the pair $(X, \|.\|)$ is called a normed space. This norm induces a metric $d\ on\ X$ by $d(x, y) = \|x - y\| \ \forall x, y \in X$.

1.8.1 Banach Space

A normed space $(X, \|.\|)$ is said to be a Banach Space if it is a complete metric space with respect to the metric induced by ||.||.

Examples:

(1) **Euclidean Space $\mathbb{R}^n$:**

We know that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with respect to the addiction and scalar multiplication defined as follow

$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

If we define $\|.\|_2$ on $\mathbb{R}^n$ by

$$\|(x_1,x_2,\dots,x_n)\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \forall (x_1,x_2,\dots,x_n) \in \mathbb{R}^n$$

Then $\|.\|_2$ is norm on $\mathbb{R}^n$

(2) **Unitary Space $\mathbb{C}^n$:**

$\mathbb{C}^n$ is a normed space with respect to the norm $\|.\|$ defined by

$$\|(z_1,z_2,\dots,z_n)\| = \left(\sum_{i=1}^{n}|z_i|^2\right)^{1/2}$$

(3) $C([a,b])$ the set of all continuous valued function on $[a,b]$

$C([a,b])$ is a vector space with respect to pointwise addition and pointwise scalar multiplication

It is a normed space with respect to the norm $\|f\| = \sup_{x\in[a,b]} |f(x)|$

Remark:

The above three normed spaces are Banach spaces but $C([a,b])$ with the norm $\|f\| = \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}}$ is a normed space but not a Banach space.

Result 1:

If $x_n \to x \text{ as } n \to \infty$ and $y_n \to y \text{ as } n \to \infty$ is a normed space $(X, \|.\|)$ then

$x_n + y_n \to x + y \text{ as } n \to \infty$ in X

(or) In a normed space X addition is continuous.

Proof:

$x_n \to x \text{ as } n \to \infty \Rightarrow \|x_n - x\| \to 0 \text{ as } n \to \infty$

$y_n \to y \text{ as } n \to \infty \Rightarrow \|y_n - y\| \to 0 \text{ as } n \to \infty$

$0 \le \|(x_n + y_n) - (x - y)\| = \|(x_n - x) + (y_n - y)\|$

$\le \|x_n - x\| + \|y_n - y\| \to 0 \text{ as } n \to \infty$

$\Rightarrow x_n + y_n \to x + y \text{ as } n \to \infty$

Result 2:

If $\alpha_n \to \alpha \; as \; n \to \infty$ in K and $x_n \to x \; as \; n \to \infty$ in X, then $\alpha x_n \to \alpha x \; as \; n \to \infty$ in X.

(or) Scalar multiplication is continuous in a normed space.

Proof:

We have $\|\alpha_n - \alpha\| \to 0 \; as \; n \to \infty$ and $\|x_n - x\| \to 0 \; as \; n \to \infty$

Now $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$

$$\le \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|$$

$$= |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|$$

$$\to |\alpha|.0 + 0.\|x\| = 0 \;\; as \;\; n \to \infty$$

$$\therefore \alpha_n x_n \to \alpha x \;\; as \; n \to \infty$$

Result 3:

Let $(X, \|.\|)$ be a normed space and d be the metric induced by $\|.\|$

$(i) \; |\|x\| - \|y\|| \le \|x - y\| \quad \forall x, y \in X$

$(ii) \; d(x + z \,, y + z) = d(x \,, y)$ (translation invariant)

$(iii) \; d(\alpha x \,, \alpha y) = |\alpha| d(x, y) \qquad \forall x, y \in X \; and \; \alpha \in K$

Proof:

(i) $\|x\| = \|x - y + y\|$

$$< \|x - y\| + \|y\|$$

$\Rightarrow \|x\| + \|y\| \le \|x - y\|$

$\Rightarrow \|y\| - \|x\| \le \|y - x\| = \|x - y\|$

$|\|x\| - \|y\|| \le \|x - y\|$

(ii) $d(x + z \,, y + z) = \|(x + z) - (y + z)\|$

$$= \|x - y\|$$

$= d(x,y)$

(iii) $d(\alpha x,\alpha y) = \|\alpha x - \alpha y\|$

$= \|\alpha(x-y)\|$

$= |\alpha|\|x-y\|$

$= |\alpha|\, d(x,y)$

Result 4:

Let $(x,\|.\|)$ be a normed space and Y be a vector subspace of X. Then $\overline{Y}$ is a vector subspace of X.

Proof:

Let $x,y \in \overline{Y}$. Then there exists a sequences (x_n) and (y_n) from Y such that

$x_n \to x$ as $n \to \infty$ and $y_n \to y$ as $n \to \infty$

Then $x_n + y_n \in Y \quad \forall n \in N$ and $x_n + y_n \to x + y$ as $n \to \infty$

$\Rightarrow x + y \in \overline{Y}$

Let $x \in \overline{Y}$ and $\alpha \in K$

Then there exists a sequence (x_n) in Y such that $x_n \to x$ as $n \to \infty$

$\alpha x_n \in Y \quad \forall n \in N$ and $\alpha x_n \to \alpha x$ as $n \to \infty$

$\Rightarrow \alpha y \in \overline{Y} \Rightarrow \overline{Y}$ is a vector space of X.

Result 5:

Let $(X_1,\|.\|_1)$ and $(X_2,\|.\|_2)$ be normed space. Then prove that $X_1 \times X_2$ is a normed space

with respect to the norm $\|(x_1,x_2)\| = max\{\|x_1\|_1,\|x_2\|_2\}$

Proof:

(i) Clearly $\|(x_1,x_2)\| = max\{\|x_1\|_1,\|x_2\|_2\} \geq 0$

$(since\ \|.\|_1,\|.\|_2\ are\ norms)$

(ii) $\|(x_1,x_2)\| = 0 \Leftrightarrow max\{\|x_1\|_1, \|x_2\|_2\} = 0$

$\Leftrightarrow \|x_1\|_1 = 0 \;\; and \;\; \|x_2\|_2 = 0$

$\Leftrightarrow x_1 = 0 \;\; and \;\; x_2 = 0$

$\Rightarrow (x_1,x_2) = (0,0)$

(iii) $\|\alpha(x_1,x_2)\| = \|(\alpha x_1, \alpha x_2)\|$

$= max\{\|\alpha x_1\|_1, \|\alpha x_2\|_2\}$

$= max\{|\alpha|\|x_1\|_1, |\alpha|\|x_2\|_2\}$

$= |\alpha| max\{\|x_1\|_1, \|x_2\|_2\}$

$= |\alpha|\|(x_1,x_2)\|$

(iv) $\|(x_1,x_2) + (y_1,y_2)\| = \|(x_1+y_1, x_2+y_2)\|$

$= max\{\|x_1+y_1\|, \|x_2+y_2\|\}$

$\leq max\{\|x_1\|_1, \|x_2\|_2\} + max\{\|y_1\|_1, \|y_2\|_2\}$

$= \|(x_1,x_2)\| + \|(y_1,y_2)\|$

$\therefore (X_1 \times X_2, \|.\|)$ is a normed space.

Result 6: Let $(X, \|.\|)$ be a normed space and Y be a closed vector subspace of X. Then $X/_Y$ is a normed space with the norm defined by $\|x+Y\|' = inf\{\|x-y\| : y \in Y\}$

Proof:

We know that $X/_Y$ is a vector space with respect to addition and scalar multiplication as follows

$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y \quad , \forall\, x_1, x_2 \in X$

$\alpha(x + Y) = (\alpha x) + Y \quad \forall x \in X, \forall \alpha \in K$

(i) $\|x+Y\|' = inf\{\|x-y\| : y \in Y\} \geq 0$

$(since\ \|x-y\| \geq 0 \;\; \forall y \in Y)$

(ii) $\|x+Y\|' = 0$

$\Rightarrow$ *there exists a sequence* (y_n) *from* Y *such that*

$\|x_n - y_n\| \to 0 \; as \; n \to \infty$

$$\Leftrightarrow\ y_n \to x \ \ as\ n \to \infty\ \ and\ y_n \in Y\ ,\ \forall n \in N$$

$$\Leftrightarrow x \in \overline{Y} = Y \quad (since\ Y\ is\ closed) \quad \Rightarrow x + Y = Y$$

(iii) $\|\alpha(x + Y)\|' = \|\alpha x + Y\|'$

$$= inf\{\|\alpha x + y\| : y \in Y\}$$

$$= inf\{\|\alpha(x - y)\| : y \in Y\}, where\ \alpha \neq 0$$

$$= inf\{|\alpha|\|x - y\| : y \in Y\} \ \ (since\ Y\ is\ vector\ space)$$

$$= |\alpha| inf\{\|x - y\| : y \in Y\}$$

$$= |\alpha|\,\|x + Y\|'$$

If $\alpha = 0$, then $\|\alpha(x + Y)\|' = \|Y\|' = 0$

$$= |\alpha|\,\|x + Y\|'$$

(iv) $\|(x_1 + Y) + (x_2 + Y)\|' = \|(x_1 + x_2) + Y\|'$

$$= inf\{\|(x_1 + x_2) - y\| : y \in Y\}$$

$$= inf\{\|(x_1 + x_2) - (y_1 + y_2)\|\}$$

$$= \ y_1 \in Y, y_2 \in Y\ (since\ Y\ is\ a\ vector\ space)$$

$$\leq inf\{\|x_1 - y_1\| + \|x_2 - y_2\| : y_1, y_2 \in Y\}$$

$$\leq inf\{\|x_1 - y_1\| : y_1 \in Y\} + inf\{\|x_2 - y_2\| : y_2 \in Y\}$$

$$= \|x_1 + Y\|' + \|x_2 + Y\|'$$

Theorem: 1.8.2

If X is a Banach space and Y is a closed subspace of X, then $X/_Y$ is a Banach space.

Proof:

By result 6, $X/_Y$ is a normed space with respect to the norm

$$\|x + Y\|' = inf\{\|x - y\| : y \in Y\}$$

Now, we prove that $X/_Y$ is complete

Let $\{x_n + Y\}$ be a Cauchy sequence in $X/_Y$

We find a subsequence $\{x_{n_k} + Y\}$ of $\{x_n + Y\}$ such that

$$\|(x_{n_k} + Y) - (x_{n_{k+1}} + Y)\|' < \frac{1}{2}k \ \forall k \in \mathbb{N}$$

By definition of $\|(x_{n_k} + Y) - (x_{n_{k+1}} + Y)\|'$ there exists y_k , y_{k+1} such that

$$\|(x_{n_k} + y_k) - (x_{n_{k+1}} + y_{k+1})\| < \frac{1}{2}k \ \forall k \in \mathbb{N}$$

If $z_k = x_{n_k} + y_k \ \forall k \in \mathbb{N}$, we claim that $\{z_k\}$ is a Cauchy sequence in X.

For $k < j$,

$$\|z_k - z_j\| = \|z_k - z_{k+1} + z_{k+1} - z_{k+2} + z_{k+2} - \cdots - z_j\|$$

$$\leq \|z_k - z_{k+1}\| + \|z_{k+1} - z_{k+2}\| + \cdots + \|z_{j+1} - z_j\|$$

$$< \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{j-1}}$$

$$< \sum_{p=k}^{\infty} \frac{1}{2^p} = \frac{1}{2^k}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2^{k-1}} \to 0 \ as \ k \to \infty$$

$\therefore \{z_k\}$ is a Cauchy sequence in X. Since X is complete $z_k \to z \ as \ n \to \infty$ for some $z \in X$.

Claim: $\{x_{n_k} + Y\}$ converges to $z + Y$ as $n \to \infty$

$$\|(x_{n_k} + Y) - (z + Y)\|' \leq \|(x_{n_k} + y_k) - z\|$$

$$= \|z_k - z\| \to 0 \ as \ n \to \omega$$

$$x_{n_k} + Y \to z + Y \quad as \ \ n \to \infty \ in \ X/_Y$$

Since $\{x_n + Y\}$ is a Cauchy sequence, and it has a convergent subsequence $\{x_{n_k} + Y\}$ is converge in $X/_Y$

$\Rightarrow X/_Y$ is complete

$\Rightarrow X/_Y$ is a Banach space.

Theorem: 1.8.3 (Completion of a normed space)

If $(X, \|.\|)$ is a normed space, which is not complete, then there exists a complete normed space $(X', \|.\|')$ such that there is a map $A: X \to X'$ $\ni:$ $\|x\| = \|A(x)\|'$, $\forall x \in X$ *and* $A(X)$ is dense in X'.

Proof:

Let d be the metric on X induced by $\|.\|$.

Then (X, d) is a metric space and hence it has a completion.

$(i.e)(X', d')$ is a complete metric space with the following properties

(i) X'= the collection of all equivalence classes of Cauchy sequences obtained by $(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0\, as\, n \to \infty$.

(ii) $d(x', y') = \lim_{n\to\infty} d(x_n, y_n)$, where $(x_n) \in x', (y_n) \in y'$

(iii)There exists a map $A: X \to X'$ such that

$d(x, y) = d'(Ax, Ay)\ \forall x, y \in X$ and $A(X)$ is dense in X'

First we define addition and scalar multiplication on X' by

$$[(x_n)] + [(y_n)] = [(x_n + y_n)]$$

$$\alpha[(x_n)] = [(\alpha x_n)]$$

Claim:

If (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ is a Cauchy sequence.

$$\|(x_n + y_n) - (x_m + y_m)\| \le \|x_n - x_m\| + \|y_n - y_m\| \to 0\ as\ m, n \to \infty$$

$\therefore$ $(x_n + y_n)$ is Cauchy

If $(x_n) \sim (x_n')$ *and* $(y_n) \sim (y_n')$ then

$\lim_{n\to\infty} \|x_n - x_n'\| = 0$ and $\lim_{n\to\infty} \|y_n - y_n'\| = 0$

Now

$$\|(x_n + y_n) - (x_n' + y_n')\| \le \|x_n - x_n'\| + \|y_n - y_n'\| \to 0\ as\ n \to \infty$$

$\therefore [(x_n + y_n)] \in X'$ and it is independent of the representatives.

Next to show that if (x_n) is a Cauchy sequence and $\alpha \in K$, then (αx_n) is a Cauchy sequence.

$\|\alpha x_n - \alpha x_m\| = \|\alpha(x_n - x_m)\| = |\alpha|\|x_n - x_m\| \to 0 \; as \; m, n \to \infty$

$\therefore (\alpha x_n)$ is Cauchy

$[(\alpha x_n)] \in X'$

The remaining properties of addition and scalar multiplication to for X' a vector space are straight forward.

Now we define $\|.\|' : X' \to \mathbb{R}$ by $\|x'\|' = d'(0', x') \;\; \forall x' \in X'$

Claim: $\|.\|'$ is a norm on X'

(i) $\|x'\|' \geq 0 \; \forall x' \in X' \, (obviously)$

(ii) $\|x'\| = 0 \Leftrightarrow d'(0', x') = 0 \Leftrightarrow x' = 0'$

(iii) Let $x' \in X' \, and \; \alpha \in K \;\; \|\alpha x'\|' = d'(0', \alpha x')$

Choose a sequence

$(x_n) \in X \;\; \ni: A \; (x_n) \to x' \;\; as \;\; n \to \infty \;\; in \; X' \;\; (since \; \overline{A(x)} = X')$

$\Rightarrow A(\alpha x_n) \to \alpha x' as \; n \to \infty \;\; in \; X'$

Since metric is continuous

$$d'(0', \alpha x') = \lim_{n\to\infty} d'(A(0), A(\alpha x_n))$$

$$= \lim_{n\to\infty} d(0, \alpha x_n)$$

$$= \lim_{n\to\infty} \|\alpha x_n\| = |\alpha| \lim_{n\to\infty} \|x_n\|$$

$$= |\alpha| \lim_{n\to\infty} d(0, x_n)$$

$$= |\alpha| \lim_{n\to\infty} d'(0, A(x_n))$$

$$= |\alpha| d'(0', x')$$

$$= |\alpha| \|x'\|'$$

(iv) Let $x', y' \in X'$. Then choose (x_n) and (y_n) in

$A(X) \ni: Ax_n \to x'$ as $n \to \infty$, $Ay_n \to y'$ as $n \to \infty$ in X'

Now $\|x' + y'\|' = d'(0', x' + y')$

$$= d'\left(A(0), \lim_{n\to\infty} A(x_n + y_n)\right)$$

$$= \lim_{n\to\infty} d'(A(0), A(x_n + y_n))$$

$$= \lim_{n\to\infty} d(0, x_n + y_n)$$

$$= \lim_{n\to\infty} \|x_n + y_n\|$$

$$\leq \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|y_n\|$$

$$= \lim_{n\to\infty} d(0, x_n) + \lim_{n\to\infty} d(0, y_n)$$

$$= \lim_{n\to\infty} d'(0', A(x_n) + \lim_{n\to\infty} d'(0', A(y_n))$$

$$= d'(0', x') + d'(0', y')$$

$$= \|x'\|' + \|y'\|'$$

$\therefore \|.\|'$ is a norm on X'.

Definition: 1.8.4

A series $\sum_{n=1}^{\infty} x_n$ in a normal linear space X is said to be

(i) Convergent if $s_n = x_1 + x_2 + \cdots + x_n \ \forall n \in \mathbb{N}$ and (s_n) converges in X.

(ii) Absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

1.8.2 Schauder Basis

A sequence (e_n) of vectors in a normed space X is said to be a schauder basis if for every $x \in X$ there exists a unique sequence of scalars (α_n) such that $\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n - x\| \to 0$ as $n \to \infty$. In other words, $x = \sum_{n=1}^{\infty} \alpha_n e_n$.

1.9 EXERCISES

(1) Let $\{e_1, \dots, e_n\}$ be a basis for a complex vector space X. Find a basis for

X regarded as a real vector space. What is the dimension of X in either case ?

(2) Let X be the vector space of all ordered pairs $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2), \ldots$ of real numbers. Show that norms on X are defined by

$$\|x\|_1 = |\xi_1| + |\xi_2|$$

$$\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2}$$

$$\|x\|_\infty = max\{|\xi_1|, |\xi_2|\}.$$

(3) Show that the closure $\bar{Y}$ of a subspace Y of a normed space X is again a vector subspace.

(4) If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

UNIT II: CONVEX SETS

Structure

2.1 INTRODUCTION

In this unit we will introduce convex set and Quotient space and also give some results. We differentiate spaces which are quotient space or not. We will discuss about the equivalent conditions of Normed spaces and some of its results. Some important results in Normed spaces are determined by theorems.

2.2 OBJECTIVES

Students will able to

- To understand the Quotient space.
- Describe the basic properties of Quotient space.
- Identify which spaces are Quotient.
- To understand the equivalent norms.
- Determine the necessary conditions of a normed space as equivalent norms.

2.3 CONVEX SETS

(a) Let V be real normed linear space and let $J: V \to \mathbb{R}$ be a given mapping. A subset K of V is said to be convex if, for every u and $v \in K$ and for every $t \in [0,1]$, we have that $tu + (1-t)v \in K$.

Let $K \subset V$ be a closed convex set. Assume that J attains its minimum over K at

$u \in K$. If J is differentiable at u, then

$J'(u)(v-u) \geq 0$ For every $v \in K$.

(b) Let K=V. If J attains its minimum at $u \in V$ and if J is differentiable at u,

Then $J'(u) = 0$.

Definition: 2.3.1

Let V be a real normed linear space. A mapping $J: V \to \mathbb{R}$ is said to be convex if,

for every u and $v \in V$ and for every $t \in [0,1]$, we have

$$J(tu + (1-t)v) \leq tJ(u) + (1-t)J(v).$$

(a) If $J: V \to \mathbb{R}$ is convex and differentiable at every point, then

$J(v) - J(u) \geq J'(u)(v-u)$. For every u and $v \in V$

(b) Let $J: V \to \mathbb{R}$ be convex and differentiable at every point of V. Let $K \subset V$ be a closed convex set. Let $u \in K$ be such that

$J'(u)(v-u) \geq 0$. For every $v \in K$, then

$J - \min_{v \in K} J(v)$.

(c) If $J: V \to \mathbb{R}$ is convex and differentiable at every point of V, and if $u \in V$ is such that $J'(u) = 0$, then J attains its minimum (over all of V) at u.

Remark:

The above definition gave the necessary conditions for a differentiable function J to attain a minimum at a point u. The preceding definition shows that these conditions are also sufficient in the case of convex functions.

Definition: 2.3.2

A norm on a vector space V is a function $\|.\| : V \to [0, \infty)$ such that

(i) $\|x\| = 0 \text{ if and only if}, x = 0;$

(ii) $\|\alpha x\| = |\alpha| \|x\| \text{ for every } \alpha \in \mathbb{F} \text{ and every } x \in V;$

(iii) $(\text{Triangle Inequality}) \text{for every } x \text{ and } y \in V, \text{we have}$

$$\|x + y\| \leq \|x\| + \|y\|.$$

2.4 NORM TOPOLOGY

A normed linear space is a vector space V endowed with a norm. The metric

Topology induced by the norm is called its norm topology.

2.4.1 Coset

Let V be a normed linear space and let W be a closed subspace of V,

$i.e.$ W is a linear subspace of V and is closed under the norm topology.

We define an equivalence relation on V by

$$x \sim y \Leftrightarrow x - y \in W.$$

The equivalence class containing a vector $x \in V$ is called a coset and is denoted as $x + W$.

It consists of all elements of the form $x + w$ where $w \in W$.

2.4.2 Quotient Space

The set of all cosets is called the quotient space and is denoted V/W.

Addition and scalar multiplication on V/W are defined by

$$(x + W) + (y + W) = (x + y) + W \text{ and } \alpha(x + W) = \alpha x + W.$$

If $x \sim x' \text{and } y \sim y'$, then, clearly, $x + y \sim x' + y' \text{ and } \alpha x \sim \alpha x'$, since

W is a linear subspace of V. Thus, addition and scalar multiplication are well defined.

Thus the quotient space becomes a vector space. On this, we define

$$\|x + W\|_{V/W} = \inf_{w \in W} \|x + w\|$$

In other words, the 'norm' defined above is the infimum of the norms of all the elements in the coset and so, clearly, it is well defined.

Theorem: 2.4.1

Let V be a normed linear space and let W be a closed subspace. Then $\|.\|_{V/W}$ defined above is a norm on the quotient space V/W. Further, if V is a Banach space, so is V/W.

Proof:

Clearly $\|x + W\|_{V/W} \geq 0$ for all $x \in V$. If $x + W = 0 + W$ in V/W, we have $x \in W$; then $-x \in W$ and so $0 \leq \|x + W\|_{V/W} \leq \|x + (-x)\| = 0$ and so $\|x + W\|_{V/W} = 0$. Conversely, if $\|x + W\|_{V/W} = 0$, then, by definition, there exists a sequence $\{w_n\}$ in W such that $\|x + w_n\| \to 0$. This means that $w_n \to -x$ in V and, since W is closed, it follows that $-x \in W$ and so $x \in W$ as well. This means that $x \sim 0$, $i.e.$ $x + W$ is the zero element of V/W.

If $\alpha \neq 0$, then $\alpha x + w = \alpha(x + w')$ where $w' = \alpha^{-1} w \in W$. From this it is easy to see that $\|\alpha x + W\|_{V/W} = |\alpha| \|x + W\|_{V/W}$. The case $\alpha = 0$ is obvious.

Finally, we prove the triangle inequality.

$$\|x + y + W\|_{V/W} = \inf\{\|x + y + w\| : w \in W\}$$

$$= \inf\{\|x + y + w + w'\| : w, w' \in W\}$$

$$\leq \inf\{\|x + w\| + \|y + w'\| : w, w' \in W\}$$

$$= \inf\{\|x + w\| : w \in W\} + \inf\{\|y + w'\| : w' \in W\}$$

$$= \|x + W\|_{V/W} + \|y + W\|_{V/W}.$$

Thus, V/W is a normed linear space. Now assume that V is complete. Let $\{x_n + W\}$ be a Cauchy sequence in V/W. Then, we can find a subsequence such that $\left\|(x_{n_k} + W) - (x_{n_{k+1}} + W)\right\|_{V/W} < \frac{1}{2^k}$

Now choose $y_k \in x_{n_k} + W$ such that $\|y_k - y_{k+1}\| < {}^1/_{2^k}$. Then the sequence

$\{y_k\}$ is Cauchy and so, since V is complete, $y_k \to y$ in V. Thus

$\left\|(x_{n_k} + W) - (y + W)\right\|_{V/W} \le \|y_k - y\| \to 0.$

Thus, the Cauchy sequence $\{x_n + W\}$ has a convergent subsequence $\{x_{n_k} + W\}$

and so the Cauchy sequence itself must be convergent and converge to the same limit.

Hence V/W is complete.

2.5 EQUIVALENT NORMS

Let X be a vector space and $\|.\|, \|.\|'$ be two norms on X. We say that these two

norms are equivalent if there exists $a > 0$ and $b > 0$ such that

$a\|x\| \le \|x\|' \le b\|x\|, \quad \forall x \in X.$

Theorem: 2.5.1

If $\|.\|$ and $\|.\|'$ are equivalent norms on X, then $A \subseteq X$ is open with respect to

$\|.\|$ if and only if $A \subseteq X$ is open with respect to $\|.\|'$

Proof:

Assume that A is open with respect to $\|.\|$.

For $x \in A$, $\exists\, r > 0 \ni: \{y \in X: \|x - y\| < r\} \subseteq A$

We claim that $\{y \in X : \|x-y\|' < ar\} \subseteq \{y \in X : \|x-y\| < r\}$

If $y \in L.H.S$, then $\|x-y\|' < ar$

$\Rightarrow \|x-y\| \leq {}^{1}/_{a}\|x-y\|' < \frac{ar}{a} = r$

$\Rightarrow y \in R.H.S$

$\Rightarrow \{y \in X : \|x-y\|' < ar\} \subseteq A$

$\Rightarrow A \text{ is open with respect to } \|.\|'$

Conversely, assume that A is open with respect to $\|.\|'$.

Then there exists $r > 0 \ \ni : \{y \in X : \|x-y\|' < r\} \subseteq A$

We claim that $\left\{y \in X : \|x-y\| < {}^{r}/_{b}\right\} \subseteq \{y \in X : \|x-y\|' < r\}$

Let $y \in L.H.S$, $\Rightarrow \|x-y\|' < {}^{r}/_{b}$

Then $\|x-y\|' \leq b\|x-y\| < {}^{r}/_{b} \times b = r$

$\Rightarrow y \in R.H.S$

$\Rightarrow A$ is open with respect to $\|.\|$.

Result:

If $\|.\|$ and $\|.\|'$ are equivalent norms on a vector space, then $\{x_n\}$ is Cauchy with respect to $\|.\|$ if and only if (x_n) is Cauchy with respect to $\|.\|'$.

Theorem: 2.5.2

Let X be a finite dimensional vector space. If $\|.\| \ and \ \|.\|'$ are norms on X, then

they are equivalent norms.

Proof:

Let $\{e_1, e_2, \dots, e_n\}$ be a basis of X.

By a theorem, there exists $c > 0$ such that

$\|\sum_{i=1}^{n} \alpha_i e_i\| \geq c \sum_{i=1}^{n} |\alpha_i|$ for all choice of α_i

Let $x \in X$ be arbitrary

Then $x = \sum_{i=1}^{n} \alpha_i e_i$ for some $\alpha_1, \alpha_2, \dots, \alpha_n \in K$

$$\|x\|' = \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|'$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \|e_i\|' \leq K \sum_{i=1}^{n} |\alpha_i| \left(\text{where } K = \max_{1 \leq i \leq n} \|e_i\|' \right)$$

$$\leq \frac{K}{c} \|\sum_{i=1}^{n} \alpha_i e_i\| = \frac{K}{c} \|x\|$$

$\therefore \|.\|$ and $\|.\|'$ are equivalent.

2.6 EXERCISES

(1) Show that the closed unit ball $\tilde{B}(0,1) = \{x \in X : \|x\| \leq 1\}$ in a normed space X is convex.

(2) Show that the norm $\|x\|$ of x is the distance from x to 0.

(3) Show that equivalent norms on a vector space X induce the same topology for X.

(4) If two norms $\|.\|$ and $\|.\|_0$ on a vector space X are equivalent, show that

$(i) \|x_n - x\| \to 0$ implies (ii) $\|x_n - x\|_0 \to 0$ (and vice versa, of course).

UNIT III: NORMED SPACES AND SUBSPACES

Structure

3.1 INTRODUCTION

Finite dimensional normed spaces and infinite dimensions spaces are important since finite dimensional spaces and subspaces play a role in various considerations (for instance, in approximation theory and spectral theory). In this unit we will introduces the most important characteristic of finite dimensional normed spaces and subspaces such as linear combinations, closedness, continuity results are introduced. Here the maximum and minimum values of a continuous mapping have been discussed.

3.2 OBJECTIVES

Students will be able to

- Identify the basic properties of finite dimensional normed space.
- Determine the compactness, completeness, continuity of a normed space in finite dimensional space.
- Recognize the difference between maximum and minimum values.

3.3 FINITE DIMENSIONAL NORMED SPACES AND SUBSPACES

Theorem 3.1: Linear combinations

Let X be a normed linear space and $\{x_1, x_2, \dots, x_n\}$ be an independent sets. Then

$\exists\, c > 0 \;\ni: \|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|)\ \forall$ choices of $\alpha_1, \alpha_2, \dots, \alpha_n \in K$.

Proof:

Let $s = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$

If $s = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ and denote the required inequality follows.

If $s > 0$, by dividing by s on both sides of the inequality, we get

$\|\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\| \geq c_1$, where $\beta_1, \beta_2, \ldots, \beta_n \in K$ with $\sum_{i=1}^{n} |\beta_i| = 1$ for some $c_1 > 0$

Suppose $inf\{\|\sum_{i=1}^{n} \beta_i x_i\| : \beta_1, \beta_2, \ldots, \beta_n \in K, \sum_{i=1}^{n} |\beta_i| = 1\} = 0$

Then there exists a sequence $(y_m = \beta_1{}^{(m)} x_1, \beta_2{}^{(m)} x_2, \ldots, \beta_n{}^{(m)} x_n)$, with

$\sum_{i=1}^{n} |\beta_i{}^{m}| = 1 \ \forall m \in \mathbb{N}$, such that $\|y_m\| \to 0 \ as\ m \to \infty$

For each $i \in \{1, 2, \ldots, n\}$, the sequence $(\beta_i{}^{(1)}, \beta_i{}^{(2)}, \ldots)$ is bounded in K. By Bolzano-Weierstrass property, $\{\beta_i{}^{(m)}\}$ has a convergent subsequence which converges to $\beta_1 (say)$. The corresponding subsequence of $\{y_m\}$ is denoted by $\{y_1, m\}$

Using $\{\beta_2{}^{(1,m)}\}$ is bounded, we find a subsequence $\{\beta_2{}^{(2,m)}\}$ of $\{\beta_2{}^{(1,m)}\}$ such that $\beta_2{}^{(2,m)} \to \beta_2 \ as\ (2, m) \to \infty$ for some β_2

Let the corresponding subsequence of $\{y_1, m\}$ be $\{y_2, m\}$.

Proceeding like this at the n^{th} stage we get a subsequence $\{y_n, m\}$ of $\{y_{n-1}, m\}$

Such that $\beta_n{}^{(n,m)} \to \beta_n$ for some $\beta_n \in K\ as\ (n, m) \to \infty$

If $y_{n,m} = \sum_{i=1}^{n} \gamma_i{}^{(m)} x_i$, then $\gamma_i{}^{(m)} \to \beta_i\ as\ m \to \infty \quad \forall i \in \{1, 2, \ldots, n\}$

Now $\{\|y_{n,m}\|\}$ is a subsequence of $\{\|y_n\|\}$,

$0 = \lim_{n \to \infty} \|y_{n,m}\| = \|y\|$, where $y = \sum_{i=1}^{n} \beta_i x_i \Rightarrow y = 0$

On the other hand, since $\sum_{i=1}^{n} |\beta_i{}^{(m)}| = 1 \ \forall i \in \{1, 2, \ldots, n\}$

$\Rightarrow \sum_{i=1}^{n} |\beta_i| = 1 \Rightarrow$not all β_i's are zeroes

$\Rightarrow$since $\{x_1, x_2, \dots, x_n\}$ is linearly independent $\Rightarrow y \neq 0$

Which is a contradiction

$inf\{\|\sum_{i=1}^{n} \beta_i x_i\| : \beta_1, \beta_2, \dots, \beta_n \in K, \sum_{i=1}^{n} |\beta_i| = 1\} = c_1$

$\Rightarrow c_1 > 0$

Hence the theorem.

Theorem: 3.2

Let X be a normed linear space. If Y is a finite dimensional subspace of X, then Y is complete.

In particular, if X is finite dimensional then X is complete.

Proof:

Let Y be a finite dimensional subspace of X with a basis $\{e_1, e_2, \dots, e_n\}$

By using theorem 3.1 there exists $c > 0$ such that

$\|\sum_{i=1}^{n} \alpha_i e_i\| \geq c \sum_{i=1}^{n} |\alpha_i| \quad \forall \alpha_1, \alpha_2, \dots, \alpha_n \in K$

Let $\{y_n\}$ be a Cauchy sequence in Y.

Let $y_m = \sum_{i=1}^{n} \alpha_i^{(m)} e_i, \ \forall m \subset \mathbb{N}$, for some suitable

$\alpha_i^{(m)} \in K, \quad i - 1, 2, \dots, n, m \in \mathbb{N}$

By definition, gives $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$\|y_m - y_r\| < \varepsilon \quad \forall m, r \geq \mathbb{N}.$

$\Rightarrow \varepsilon > \|y_m - y_r\| = \|\sum_{i=1}^{n} \alpha_i^{(m)} e_i - \sum_{i=1}^{n} \alpha_i^{(r)} e_i\|$

$= \|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(r)}) e_i\|$

$\geq c \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}| \geq |\alpha_i^{(m)} - \alpha_i^{(r)}|$

$\forall i = 1, 2, \dots, n$

$\{\alpha_i^{(m)}\}$ is a Cauchy sequence in K $\forall i = 1, 2, \dots, n$

Since $K(=\mathbb{R}\ or\ \mathbb{C})$ is complete,

$\alpha_i^{(m)} \to \alpha_i\ as\ m \to \infty\ \ \forall i \in \{1,2,\dots,n\}$

If $y = \sum_{i=1}^{n} \alpha_i e_i$, then

$\|y_m - y\| = \|\sum_{i=1}^{n} \alpha_i^{(m)} e_i - \sum_{i=1}^{n} \alpha_i e_i\|$

$= \|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i) e_i\|$

$\leq \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i| \|e_i\| \to 0\ as\ m \to \infty$

$\Rightarrow y_m \to y \quad as\ m \to \infty$

∴ Y is complete.

Theorem: 3.3

Y is a finite dimensional subspace of a normed linear space X, then Y is closed in X.

Proof:

Let $x \in X$ be a limit point of Y.

Then there exists a sequence $\{y_m\}$ from Y, such that $y_m \to x\ as\ n \to \infty$ in X.

⇒$\{y_m\}$ is Cauchy in Y

By theorem 3.2, Y is complete

Therefore, $\{y_m\}$ converges to a point in Y

Since $\lim_{n\to\infty} y_n = x$, we have $x \in Y$

Y contains all of its limit points

Y is closed in X.

3.4 COMPACTNESS

A metric space M is said to be compact of every sequence (x_n) from M has a convergent subsequence.

Theorem: 3.4

A subset K of a metric space is compact, then K is closed and bounded.

Proof:

Let x be a limit point of K

Then there exists a sequence (x_n) from K converges to x

Then by compactness $x \in K$.

(since (x_n) has a convergent subsequence whose limit $x \in K$)

Suppose K is not bounded, for a fixed $y \in K$, and for each

$n \in \mathbb{N}$ there exists $x_n \in K$ such that $d(x_n, y) > n \quad \forall n \in \mathbb{N}$

$\Rightarrow (x_n)$ has no convergent subsequence

(since every subsequence of (x_n) is unbounded and hence not convergent)

⇒K is not compact which is a ⇒⇐

⇒K is bounded.

Remark:

Converse of this theorem is not true. Consider $(\mathbb{N}, d)$, where $\mathbb{N}$ is the set of all

natural numbers and d is the discrete metric. Then $\mathbb{N}$ is closed and bounded

but not compact.

Theorem: 3.5

Let X be a finite dimensional normed space and $A \subseteq X$. A is compact if and only if A is closed and bounded.

Proof:

By theorem 3.4, if A is compact then it is closed and bounded.

Conversely, assume that A is closed and bounded.

To prove: A is compact

Let (x_n) be a sequence from A

$\Longrightarrow \{x_n\}$ is bounded (since A is bounded)

$\Longrightarrow \exists\, M > 0 \ni : \|x_n\| \le M \quad \forall x \in \mathbb{N}$

Let $\{e_1, e_2, \dots, e_m\}$ be a basis of X.

Then $x_n = \sum_{i=1}^{m} \alpha_i^{(n)} e_i$ for some suitable scalars $\alpha_i^{(n)}$,$\forall n \in \mathbb{N}$

By a theorem,

$\exists\, c > 0 \ni : \|x_n\| = \left\|\sum_{i=1}^{m} \alpha_i^{(n)} e_i\right\| \ge c \sum_{i=1}^{m} \left|\alpha_i^{(n)}\right| \quad \forall n \in \mathbb{N}$

$\Longrightarrow$ for each $i \in \{1, 2, \dots, m\}$, $\{\alpha_i^{(n)}\}$ is a bounded sequence.

Then there exists a subsequence (z_n) of (x_n) $\ni : z_n = \sum_{i=1}^{m} \beta_i^{(n)} e_i$ and

$\beta_i^{(n)} \to \beta_i \quad as\ n \to \infty \quad \forall\, i = 1, 2, \dots, m$

$\Longrightarrow$ If $z = \sum_{i=1}^{m} \beta_i e_i$, then $\|z_n - z\| \le \sum_{i=1}^{m} \left|\beta_i^{(n)} - \beta_i\right| \|e_i\| \to 0 \quad as\ n \to \infty$

$\Longrightarrow (x_n)$ has a convergent subsequence.

$\Longrightarrow$ A is compact (since A is closed, z∈A)

Lemma: 3.6 (F.Riesz's)

Let X be a normed space and Y, Z be subspaces of X. If Y is closed and Y is a

Proper subspace of Z, then for every $\theta \in (0,1) \quad \exists\, z \in Z \quad \ni : \|z\| = 1$ and

$\|y - z\| \ge \theta \quad \forall\, y \in Y$

Proof:

Let $v \in Z/Y$

If $a = inf\{\|v - y\| : y \in Y\}$, then $a > 0 \quad (since\ Y\ is\ closed)$

Self-Instructional Material

Let $\theta \in (0,1)$ be arbitrary.

$\Rightarrow \frac{a}{\theta} > a$

Since $a = inf\{\|v - y\| : y \in Y\}$ $\exists\, y_0 \in U$ $\ni: a < \|v - y_0\| < {}^{a}/_{\theta}$

Let $z = c(v - y_0)$, $where\ c = \frac{1}{\|v-y_0\|}$

$\Rightarrow z \in Z\ and\ \|z\| = 1$

Claim: $\|y - z\| \geq \theta \quad \forall y \in Y$

$\|y - z\| = \|y - c(v - y_0)\|$

$= c\|c^{-1}y - v + y_0\|$

$= c\|v - (c^{-1}y + y_0)\|$

$\geq ca \quad (since\ c^{-1}y + y_0 \in Y)$

$= \frac{a}{\|v-y_0\|} \geq \frac{a}{{}^{a}/_{\theta}} = \theta$

Theorem: 3.7

If X is a normed space such that $M = \{x \in X : \|x\| \leq 1\}$ is compact, then X is finite dimensional

Proof:

Suppose dim X $= +\infty$

Fix $x_1 \in M$ be arbitrary such that $x_1 \neq 0$

Then the subspace generated by x_1 is a closed proper subspace of X.

Then by Riesz's lemma, there exists $x_2 \in X$ $\ni: \|x_2\| = 1\ and$

$\|x_2 - x_1\| \geq {}^{1}/_{2}$

Since the subspace generated by $\{x_1, x_2\}$ is closed and properly contained in

X, there exists $x_3 \in X$ $\ni: \|x_3\| = 1\ \ and\ \|x_3 - x_i\| \geq {}^{1}/_{2}\, , i = 1,2, \ldots$

Proceeding like this, we find a sequence (x_n) from X such that $\|x_n\| = 1$

$and\ \|x_n - x_i\| \geq {}^1/_2 \quad \forall\, i = 1,2,\dots,n-1, \quad \forall\, n \in \mathbb{N}$

We note that (x_n) is a sequence in M such that it has no convergent

Subsequence

$\Longrightarrow$ M is not compact

Which is a $\Rightarrow\Leftarrow$

X has finite dimension.

Theorem: 3.8

Let X and Y be metric spaces and $T: X \to Y$ a continuous mapping. Then the image of a compact subset M of X under T is compact.

Proof:

By the definition of compactness it suffices to show that every sequence (y_n)

in the image $T(M) \subset Y$ contains a subsequence which converges in $T(M)$,

we have $y_n = Tx_n$ for some $x_n \in M$. Since M is compact, (x_n) contains a

subsequence (x_{n_k}) which converges in M. The image of (x_{n_k}) is a subsequence of (y_n) which converges in $T(M)$ by knowing theorem (A mapping $T: X \to Y$ of a Metric space (X, d) into a metric space (Y, d) is continuous at a point $x_0 \in X$ iff $x_n \to x_0$ implies $Tx_n \to Tx_0$) because T is continuous. Hence $T(M)$ is compact.

Corollary: (Maximum and minimum)

A continuous mapping T of a compact subset M of a metric space X into $\mathbb{R}$ assumes a maximum and a minimum at some points of M.

Proof:

$T(M) \subset \mathbb{R}$ is compact by above theorem and closed and bounded by theorem 3.3 [applied to $T(M)$], so that $infT(M) \in T(M)$, $supT(M)$

$\in T(M)$, and the inverse images of these two points consist of points of M at which T_x is minimum or maximum, respectively.

3.5 EXERCISES

(1) Show that $\mathbb{R}^n$ and $\mathbb{C}^n$ are not compact

(2) Show that a discrete metric space X consisting of infinitely many points is not compact.

(3) Give examples of compact and non compact curves in the plane $\mathbb{R}^2$.

(4) If $\dim Y < \infty$ in F.Riesz's theorem , show that one can even choose $\theta = 1$.

UNIT IV: LINEAR OPERATOR

Structure

4.1 INTRODUCTION

In calculus we consider the real-valued functions on R(or on a subset of R). obviously, any such is a mapping of its domain in to R. In functional analysis we consider the more general spaces such as metric spaces and normed spaces and mapping of their spaces.The mappings of such spaces known as operator and an operator is generally a mapping that acts on a elements of a space to produce elements of another space. In this chapter the abstract idea of linear operators and bounded linear operators have been discussed.

4.2 OBJECTIVES

The students will be able to,

- Analyse the concepts of Range space and null space of linear operators
- The existence of inverse operator of T
- Continuity and boundedness of linear operators

4.3 LINEAR OPERATOR

Definition: 4.1 A linear operator T is an operator such that

(i) The domain D(T) of T is a vector space and the range R(T) lies in a vector space over the same field

(ii) For all $x, y \in D(T)$ and scalars α, it satisfies the following

$$T(x+y) = Tx + Ty$$

$$T(\alpha x) = \alpha Tx$$

Note: 4.1.2 D(T) denotes the domain of T

R(T) denotes the range of T

N(T) denotes the null space of T

Definition: 4.1.3 The null space of T is the set of all $x \in D(T)$ *such that* $T(x) = 0$

Example: 4.1.4

1. The identity operator $I: X \to X$, defined by $I(x) = x \ \forall \ x \in X$, is bounded and $\|I\| = 1$
2. The zero operator defined by $0: X \to X \forall \ x \in X$, is bounded and $\|0\| = 0$
3. Differentiation: Let X be the vector space of all polynomials on [a,b]. we may define a linear operator T on X by setting $Tx(t) = x^{I}(t)$ *for every* $x \in X$, where the prime denotes differentiation with respect to t . This operator T maps X onto itself.

Theorem: 4.1 (Range and null space). Let T be a linear operator. Then:

(a) The range R(T) is a vector space.

(b) If dim $D(T) = n < \infty$, *then* $dim R(T) \leq n$.

(c) The null space N(T) is a vector space.

Proof: (a) We take any $y_1, y_2 \in R(T)$

To prove: $\alpha y_1 + \beta y_2 \in R(T)$ for any scalars α, β

Let $y_1, y_2 \in R(T)$,

we have $y_1 = Tx_1, y_2 = Tx_2$ *for some* $x_1, x_2 \in D(T)$ and

$\alpha x_1 + \beta x_2 \in D(T)$ because D(T) is a vector space.

The linearity of T yields,

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2$$

Hence $\alpha y_1 + \beta y_2 \in R(T)$.

$since\ y_1, y_2 \in R(T),$ were arbitrary.

This proves that R(T) is a vector space.

(b) We choose n+1 elements $y_1 \dots \dots y_n\ of\ R(T)$ in an arbitrary manner.

Then we have,

$y_1 = Tx_1, \dots, y_{n+1} = Tx_{n+1}\ \ for\ some\ x_1, x_2, \dots, x_{n+1}\ in\ D(T).$

$since$ dim $D(T) = n,$

$This\ set\ \{x_1, x_2, \dots, x_{n+1}\}$ must be linearly dependent.

Hence, $\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$

For some scalars not all zero.

Since T is linear and T0=0, application of T on both sides

$T(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + \cdots + \alpha_{n+1} y_{n+1} = 0$

This shows that $\{y_1, y_2, \dots, y_{n+1}\}$ is a linearly dependent set because not all scalars are zero. Since the subset R(T) was chosen in an arbitrary manner, we conclude that R(T) has no linearly independent subsets of n+1. $\Rightarrow dim R(T) \leq n$

(c) We take any $x_1, x_2 \in N(T)$

Then $Tx_1 = Tx_2 = 0.$

Since T is linear, for any scalar α, β we have

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = 0.$$

This shows that $(\alpha x_1 + \beta x_2) \in N(T)$ is a vector space.

Theorem: 4.2 (Inverse operator). Let X, Y be vector spaces, both real or both complex. Let $T: D(T) \to Y$ be a linear operator with domain $D(T) subset\ of\ X$ and range R(T) subset of Y. Then

(a) The inverse $T^{-1}: R(T) \to D(T)$ exists iff Tx=0 implies x=0

(b) If T^{-1} exists, it is a linear operator.

(c) If dim $D(T) = n < \infty, and\ T^{-1}$ exists, then $\dim R(T) = \dim D(T)$.

Proof: (a) Suppose that Tx=0 implies x=0.

Let $Tx_1 = Tx_2$, Since T is linear,

$T(x_1 - x_2) = Tx_1 - Tx_2 = 0,$

So that $x_1 - x_2 = 0$ (by the hypothesis)

Hence $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$

T^{-1}exists.

Conversely if T^{-1} exists with $x_2 = 0$ and we obtain,

$Tx_1 = T0 = 0 \Rightarrow x_1 = 0$

This completes the proof of (a).

(b) We assume that T^{-1} exists

To prove: $T^{-1}$is linear.

The domain of T^{-1} is R(T) and is a vector space (by theorem 4.1 (a)). We consider any $x_1, x_2 \in D(T)$ and their images

$y_1 = Tx_1 \quad and y_2 = Tx_2$

Then $x_1 = T^{-1}y_1 \quad and x_2 = T^{-1}y_2$

T is linear, so that for any scalars $\alpha\ and\ \beta$ we have,

$T(\alpha y_1 + \beta y_2) = \alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2)$s

Since, $x_j = T^{-1}y_j, this\ implies$

$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = (\alpha T^{-1}y_1 + \beta T^{-1}y_2)$

And proves that T^{-1} is linear.

(c) we have , $\dim R(T) \le \dim D(T)$(by Theorem:4.1(a))

$\dim D(T) \le \dim R(T)$by the same theorem

applied to T^{-1}.

Lemma: 4.3 (Inverse of product). Let $T: X \to Y$ *and* $S: Y \to Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then the inverse $(ST)^{-1}: Z \to X$ of the product(the composite) ST exists and $(ST)^{-1} = T^{-1}S^{-1}$

Proof: The operator $ST: X \to Z$ is bijective, so that $(ST)^{-1}$ exists. We thus have

$ST(ST)^{-1} = I$

Where I is the identity operator on Z.

Applying S^{-1} and using $S^{-1}S = I$(the identity operator on Y), we obtain,

$S^{-1}ST(ST)^{-1} = T(ST)^{-1} = S^{-1}I = S^{-1}$.

Applying $T^{-1}$and using$T^{-1}T = I$, we obtain the desired result.

$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1}$

This completes the proof.

4.4 BOUNDED LINEAR OPERATOR

Definition: 4.1.5 Let X and Y be normed spaces D(T) be a subspace of X. A linear operator $T: D(T) \to Y$ is called a bounded linear operator if $\exists\, c > 0$, such that $\|Tx\| \leq c\|x\| \;\forall\, x \in D(T)$

Definition: 4.1.6 Let T be a bounded linear operator on D(T). Then norm of T is defined by, $\|T\| = sup_{0 \neq x \in D(T)} \frac{\|Tx\|}{\|x\|}$

Remark: 4.1.7 For every bounded linear operator T, $\|T\| \leq +\infty$

Example: 4.1.8

4. The identity operator $I: X \to X$, defined by $I(x) = x \;\forall\, x \in X$, is bounded and $\|I\| = 1$

5. The zero operator defined by $0: X \to X \forall\, x \in X$, is bounded and $\|0\| = 0$

6. If $A = (\alpha_{ij})_{n\times n}$ is a matrix then A defines a linear operator on R^n in to R^r by $y = Ax,$ where $y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_r \end{pmatrix} \in R^r$ and $x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in R^n$ Then this operator is bounded .

$$\|Ax\|^2 = \|y\|^2 = \sum_{i=1}^{r} y_i^2 = \sum_{i=1}^{r}\left(\sum_{i=1}^{n} \alpha_{ik}x_k\right)^2$$

$$\leq \sum_{i=1}^{r}\left(\sum_{k=1}^{n} \alpha_{ik}^2\right)\left(\sum_{k=1}^{n} x_k^2\right)$$

$$= c^2\|x\|^2,$$

where $c^2 = \sum_{i=1}^{r}\sum_{k=1}^{n} \alpha_{ik}^2$

Then $\|Ax\| \leq C\|x\|$

Since x is arbitrary, A is bounded.

7. The differential operator from the normed space of polynomials on $[a.b]$ is not bounded. Because if $p_n(t) = t^n \ \forall\, t \in [a.b]$ then

$$\|DP_n\| = \|np_{n-1}\| = \sup|nt^{n-1}| \quad t \in [a.b]$$

$$= |n|b^{n-1}$$

If $b \geq 1,\ \|DP_n\| \to \infty \ as\ n \to \infty$

Therefore D is not bounded.

Result: 4.1.9 If T is a bounded linear operator on D(T) then, $\|T\| = sup_{\substack{x\in D(T) \\ \|x\|=1}}\|Tx\|$

Proof: Clearly, $sup_{\substack{x\in D(T) \\ \|x\|=1}}\|Tx\| \leq \ \|T\| = sup_{0\neq x\in D(T)} \frac{\|Tx\|}{\|x\|}$

Let $x_o \in D(T)\ with\ x_o \neq 0$

Then let, $y = \frac{x_o}{\|x_o\|}$

$$\frac{\|Tx_o\|}{\|x_o\|} = \|Ty\| \le sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$$

$$\|T\| = sup_{0 \neq x \in D(T)} \frac{\|Tx_o\|}{\|x_o\|} \le sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$$

$$\|T\| = sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$$

Then $\exists, c > 0, \ni: \|Tx\| \le c\|x\|, \forall x \in D(T)$

Example: 4.1.10 If T is a bounded linear operator from D(T) in to itself then $\|T^n\| \le \|T\|^n \ \forall n \in N$

Proof: First we prove that if $T_1: X \to Y$ and $T_2: Y \to Z$ are bounded linear operators then $\|T_2T_1\| \le \|T_2\|\|T_1\|$. Since T_2 is bounded,

$\|T_2(T_1(x))\| \le \|T_2\|\|T_1x\| \ \forall x \in X$

$\le \|T_2\|\|T_1\|\|x\|$

$\|T_2\|\|T_1\|$is an upper bound for $\left\{\frac{\|T_2T_1x\|}{\|x\|} : x \in X, \ x \neq 0\right\}$

$\|T_2T_1\| \le \|T_2\|\|T_1\|$

By taking $T_1 = T_2 = T$, we get

$\|T^2\| \le \|T\|^2$

By induction we get

$\|T^n\| \le \|T\|^n \quad \forall n \in N$

Example: 4.1.11 If T is a bounded linear operator from D(T) in to Y, then

(i) $x_n \to x as n \to \infty in D(T) implies Tx_n \to Tx as n \to \infty in Y$

(ii) N(T) is closed in D(T)

Proof: (i) By theorem (4.6) T is bounded iff T if continuous

Therefore (i) follows immediately.

(ii) $N(T) = \{x \in D(T): Tx = 0\}$

$= \{T^{-1}\{0\}\}$

Since $\{0\}$ is closed in Y and T is continuous we have,

$T^{-1}(\{0\}) = N(T) is closed$

Example: 4.1.12

Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive $b \ni: \|Tx\| \geq b\|x\| \ \forall x \in X,$ then $T^{-1}: y \to x$ exists and T^{-1} is a bounded linear operator.

Proof: If $x \in X \ and \ Tx = 0, then\ 0 = \|Tx\| \geq b\|x\|$

$\|x\| \leq 0 \Longrightarrow \|x\| = 0 \Longrightarrow x = 0$

$\Longrightarrow T \ is \ one - one$

$$\Longrightarrow T^{-1}: Y \longrightarrow X \ exists \ (T \ is \ a \ bijection)$$

We know that T^{-1} exists. Let $y \in Y$ be arbitrary,

Since, $T: X \to Y$ is onto, $\exists \ x \in X \ \ni: Tx = y$

$\|T^{-1}y\| = \|T^{-1}Tx\| = \|x\| \leq \frac{1}{b}\|y\|$

$$\Rightarrow \|T^{-1}\| \leq \frac{1}{b} < \infty$$

$\Rightarrow T^{-1} \ is \ bounded.$

4.5 EXERCISE

1. Let $T: X \to Y$ be a linear operator. Show that the image of a subspace of V of X is a vector space, and so is the inverse image of a subspace W of Y
2. If the product of two linear operators exists, show that it is linear.
3. Let X and Y be normed spaces show that a linear operator $T: X \to Y$ is bounded iff T maps bounded sets in X in to bounded sets in Y.

4. Show that the range R(T) of a bounded linear operator $T: X \to Y$ Need not to be closed in Y
5. Let $T: X \to Y$ be a linear operator and dimX= dimY= n. show that R(T)=Y iff T^{-1} exists.

UNIT V: LINEAR FUNCTIONAL

Structure

5.1 INTRODUCTION

Functional analysis was initially the analysis of functionals. A functional is an operator whose range lies on the real or complex plane. Functionals are operators so the previous definitions apply further our goal is to show, if we take any two normed spaces X and Y and consider the set B(X,Y) consisting of all bounded linear operators from X in to Y. we will prove that B(X, Y) itself made in to a normed space.

5.2 OBJECTIVES

The Students will be able to,

- Analyse the concepts of linear functionals and bounded linear functionals
- Algebraic dual and canonical mapping of spaces and oriented results
- Determine the completeness of B(X,Y)

5.3 LINEAR FUNCTIONALS

Definition: 5.1.1 A linear functional f is a linear operator with domain in a vector space X and range in the scalar field K of X; thus $f: D(f) \to K$ Where K=R if X is real and K=C if X is complex.

Example: 5.1.2

The norm $\|.\| : X \to R$ on a normed space (X, $\|.\|$) is a functional on X which is not linear.

Space C[a, b] is a functional on X.

Definition: 5.1.3 A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $D(f)$ lies. Thus there exist a real number c such that for all $x \in D(f)$, $|f(c)| \leq c\|x\|$.

For a continuous linear functional f on D(f) the norm $\|f\|$ *of* f is defined by,

$$\|f\| - sup_{0 \neq x \in D(f)} \frac{|f(x)|}{\|x\|}$$

Example: 5.1.4 Consider for each $a \in R^3, f_a: R^3 \to R$ is defined by

$f_a(x) = a.x \; \forall \; x \in R^3$.

Claim: f_a is linear

Let $x, y \in R^3$,

$$f_a(x + y) = f_a((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$= f_a(x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$= a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3)$$

$$= a_1x_1 + a_1y_1 + a_1x_2 + a_1y_2 + a_1x_3 + a_1y_3$$

$$= (a_1x_1 + a_1x_2 + a_1x_3) + (a_1y_1 + a_1y_2 + a_1y_3)$$

$$= a(x) + a(y)$$

$$= f_a(x) + f_a(y)$$

Let $x \in R^3$ *and* $\alpha \in R$,

$$f_a(\alpha x) = f_a(\alpha x_1 + \alpha x_2 + \alpha x_3)$$

$$= (a_1, a_2, a_3)(\alpha x_1, \alpha x_2, \alpha x_3)$$

$$= a_1(\alpha x_1) + a_2(\alpha x_2) + a_3(\alpha x_3)$$

$$= \alpha(a.x)$$

$$= \alpha(f_a(x))$$

Claim: $\|f_a\| = \|a\|$

For every $x \in R^3$

$|f_a(x)| = |a.x| \leq \|a\|\|x\|$ $(By schwartz inequality$

$$\|f_a\| \leq \|a\|$$

If $a \neq 0, let\ x = \frac{a}{\|a\|}$

Then $\|f_a\| \geq |f_a(x)| = |a.x|$

$$= \frac{|a a|}{\|a\|} = \|a\|$$

$$\|f_a\| \geq \|a\|$$

Thus $\|f_a\| = \|a\|$

If $a = 0, then\ f_a = 0\ implies\ \|f_a\| = \|0\| = 0$

Example: 5.1.5 Let $C([a,b])$ be the Banach space of all real valued continuous functions on $[a,b]$ with the supremum norm. clearly the integral operator f is a linear functional on $C([a,b])$. We recall that f is defined by,

$$f(x) = \int_a^b x(t)dt, \forall x \in C([a,b])$$

Claim: $\|f\| = b - a$

$$|f(x)| = \left|\int_a^b x(t)dt\right|$$

$$\leq \int_a^b |x(t)|dt$$

$$\leq sup_{t\in[a,b]}|x(t)| \int_a^b dt$$

$$= \|x\|(b-a)$$

$\|f\| \leq (b-a)$

Clearly, the constant function 1 $\in C([a,b])$.

$\|f\| \geq |f(1)|$

$$= \left|\int_a^b 1.dt\right|$$

$= |b-a|$

$\|f\| \geq |b-a|$

It implies that, $\|f\| = b-a$

Definition: 5.1.6 Let X be a vector space. The space of all linear functionals on X is called the algebraic dual of X and it is denoted by X^*.

$X^* = \{f: X \to K : f\ is\ linear\}$

Clearly $X^*$is a vector space with respect to pointwise addition and scalar multiplication.

Definition: 5.1.7 The map $C: X \to X^{**}$ is called the canonical map if C is defined by $c(x) = f_x\ \forall x \in X$ where $f_x: X^* \to K$ defined by $f_x(g) = g(x)\ \forall g \in X^*$

Lemma: 5.1 For every $x \in X, f_x \in X^{**}$ (or) equivalently $C: X \to X^{**}$

Proof: Fix $x \in X$ be arbitrary.

To prove: $f_x \in X^{**}$

we shall show that f_x is a linear functional on X^*.

Clearly, for every $g \in x^*$, $f_x(g) = g(x) \in K$.

It implies that $f_x$is a functional on X^*.

Let $g_1, g_2 \in x^*$ Then,

$f_x(g_1 + g_2) = (g_1 + g_2)x$

$$= (g_1(x) + g_2(x))$$

$$= f_x(g_1) + f_x(g_2)$$

Let $\alpha \in K$ and $g \in X^*$ Then,

$$f_x(\alpha g) = (\alpha g)(x)$$

$$= \alpha(g(x))$$

$$= \alpha f_x(g)$$

Therefore, f_x is a linear functional on X^*.

$$\Rightarrow f_x \in X^{**}$$

$$\Rightarrow C : X \to X^{**}$$

Definition: 5.1.8 A vector space X is said to be algebraically reflexive if X is isomorphic on to X^{**},

(i.e.,) $C(X) = X^{**}$

Result: 5.1.9 $C : X \to X^{**}$ is linear and one to one.

Proof: Let $x_1, x_2 \in X$

Claim: $C(x_1 + x_2) = C(x_1) + C(x_2)$

$f_{x_1+x_2} = f_{x_1} + f_{x_2}$ on X^*

Let $g \in X^*$ be arbitrary. Then,

$$f_{x_1+x_2}(g) = (x_1 + x_2)g$$

$$= g(x_1) + g(x_2)$$

$$= f_{x_1}(g) + f_{x_2}(g)$$

$$= f_{x_1} + f_{x_2}(g)$$

Claim: For $\alpha \in K$ and $x \in X$,

$$C(\alpha x) = \alpha(Cx)$$

Let $g \in X^*$ be arbitrary

$f_{\alpha x}(g) = (g)(\alpha x)$

$= \alpha(g(x))$

$= \alpha f_x(g)$

Hence C is linear.

Claim: C is one to one

We shall show that if $x \in X \ \ni c(x) = 0 \ then \ x = o$

$$c(x) = 0$$

$\Rightarrow f_x = 0 \ on \ X^*$

$$\Rightarrow f_x(g) = 0 \quad \text{for every } g \in X^*$$

$$\Rightarrow g(x) = 0 \quad \text{for every } g \in X^*$$

Claim: $x = o$

Let $\{v_\alpha\}$ be a basis of x.

Since, $x \in X,\ \ x = \sum_{i=1}^{n} c_i v_{\alpha_i} \ for \ some \ v_{\alpha_i}$

If we define a linear map g_i on X, such that

$$g_i(v_\alpha) = \begin{cases} 1 \ if \alpha = \alpha_i \\ 0 \ if \alpha \neq \alpha_i \end{cases}$$

By assumption we have $g_i(x) = 0 \ \ for \ every \ i = 1,2 \dots n$

$0 = g_i(x) = g_j(\sum_{j=1}^{n} c_j v_{\alpha_j})$

$$= \sum_{j=1}^{n} c_j v_{\alpha_j}$$

$= c_j \ \ for \ every \ i = 1,2, \dots n$

$\Rightarrow x = 0$

Therefore c is one to one.

Theorem: 5.2 If X is finite dimensional vector space then $\dim X = \dim X^*$

Proof: Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of X.

For each i=1,2,..n we define,

$$f_i(e_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Thus $\{f_1, f, \ldots f_n\}$ $subset\ of\ X^*$

Claim: $\{f_1, f, \ldots f_n\}$ is linearly independent.

If $\sum_{i=1}^{n} \alpha_i f_i = 0 \quad for\ some \quad \alpha_1, \alpha_2, \ldots \alpha_n$ in K, Then

Since $e_j \in X, for\ every\ j = 1,2 \ldots n$

$$\sum_{i=1}^{n} \alpha_i f_i(e_j) = 0 \ \ forevery j = 1,2 \ldots n$$

$$\Rightarrow \alpha_i = 0$$

$\rightarrow \{f_1, f, \ldots f_n\}$is linearly independent.

Claim: $\{f_1, f, \ldots f_n\}$ spans X^*

Let $\quad f \in X^*\ be\ arbitary,$

We claim that $f = \sum_{i=1}^{n} f(e_i) f_i$

Let $x \in X$ be arbitrary,

Then $x = \sum_{i=1}^{n} \beta_i e_i = 0 \quad for\ some\ \beta_i$

$$f(x) = f(\sum_{i=1}^{n} \beta_i e_i) = \sum_{i=1}^{n} \beta_i f(e_i))$$

$$\Rightarrow f - \sum_{i=1}^{n} f(e_i) f_i$$

$\therefore \{f_1, f, \ldots f_n\}$ $is\ a\ basis\ of\ X^*$

$\Rightarrow \dim X = \dim X^* = n$

Corollary: 5.3 If X is a finite dimensional vector space then X is algebraically reflexive.

Proof: We know that the canonical mapping $C: X \to X^{**}$ is linear and one to one

By theorem 5.9 $\dim X = \dim X^{*} = \dim X^{**}$

Clearly, $C: X \to R(C)$ is an isomorphism.

$\Longrightarrow \dim X = \dim R(C)$

But R(C) is a subspace of X^{**}

$\Longrightarrow R(c) = X^{**}$

$\Longrightarrow C \text{ is on to}$

$\Longrightarrow C \text{ is algebrically reflextive.}$

5.4 NORMED SPACES OF OPERATORS

Theorem: 5.4 The space of all bounded linear operators from X in to Y B(X,Y) is a normed space.

Proof: Firstly, B(X,Y) is a vector space if we define the addition as,

$(T_1 + T_2)x = T_1x + T_2x$

And scalar multiplication defined as,

$(\alpha T)x = \alpha Tx$

Claim: B(X,Y) is a normed space

Norm is defined as, $\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$

We know that the norm (N1) is obvious from the definition and

$\|0\| = 0, \quad from \|T\| = 0$

We have $Tx = 0 \quad \forall\, x \in D(T), so\ that \quad T = 0$

Hence (N2) holds, (N3) is obtained from,

$$sup_{\|x\|=1}\|\alpha Tx\| = sup_{\|x\|=1}|\alpha|\|Tx\|$$

$$= |\alpha| sup_{\|x\|=1}\|Tx\|$$

Where, $x \in D(T)$, finally (N4) follows from

$$sup_{\|x\|=1}\|(T_1+T_2)x\| = sup_{\|x\|=1}\|T_1x+T_2x\|$$

$$\leq sup_{\|x\|=1}\|T_1x\| + \|T_2x\|$$

Hence B(X,Y) is a normed space.

Theorem:5.5 If X is a normed space and Y is a banach space, Then B(X,Y) is a banach space.

Proof: We know that B(X,Y) is a banach space.

To prove: B(X, Y) is a banach space. We show that $\{T_n\}$ is a a cauchy sequence in B(X, Y), then $T_n \to T \;\; as\, n \to \infty \; for\; some\; T \in B(X,Y).$

Claim: For each $x \in X, \{T_nx\}$ is a Cauchy sequence in Y.

For m, n $\in$N,

$$\|T_nx - T_mx\| = \|(T_n - T_m)x\|$$

$$\leq \|T_n - T_m\|\|x\| \to 0$$

as $m, n \to \infty$ (since,$\{T_n\}$ is a a cauchy sequence in B(X,Y))

implies that, $\{T_nx\}$ is a a cauchy sequence in Y.

Since Y is complete, for each x $\exists\, y_x \in Y$ Such that, $T_nx \to y_x \;\; as \;\; n \to \infty$

Define: $T: X \to Y\; by\; T(x) = y_x \;\; \forall\; x \in X$

Let $x_1, x_2 \;\in X$

$$T(x_1+x_2) = \lim_{n\to\infty} T_n(x_1+x_2)$$

$$= \lim_{n\to\infty}(T_n(x_1)+T_n(x_2))$$

$$= \lim_{n\to\infty}(T_n(x_1)) + \lim_{n\to\infty}(T_n(x_2))$$

$$= Tx_1 + Tx_2$$

Let $x \in X$ and $\alpha \in K$,

$$T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x)$$

$$= \lim_{n\to\infty} \alpha T_n(x)$$

$$= \alpha \lim_{n\to\infty} T_n(x)$$

$$= \alpha T x$$

Thus, T is a linear operator.

Applying $m \to \infty$ and keeping n fixed, in

$$\|T_n x - T_m x\| \le \|T_n - T_m\| \|x\|$$

Implies that $T_n - T \in B(X,Y)$

Therefore B(X,Y) is complete.

5.5 EXERCISE

1. If f is a bounded linear functional on a complex normed space, is $\bar{f}$ bounded? Linear? (The bar denotes the complex conjugate)
2. Show that two linear functionals $f_1 \neq 0$ and $f_2 \neq 0$ which are defined on the same vector space and have the same null space are proportional.
3. Show that the functionals defined on C[a,b] by $f_2(x) = \alpha x(a) + \beta x(b)$ are linear and bounded.
4. What is the zero element of the vector space B(X,Y)? The inverse of a T belongs to B(X,Y)?
5. Show that a linear functional f on a vector space X is uniquely determined by its values on a hamel basis for X.

UNIT VI: BOUNDED OR CONTINUOUS LINEAR OPERATOR

Structure

6.1 INTRODUCTION

In this section the most important characteristic of linear operators as such continuity, boundedness and its related results are introduced and for any vector space it has its corresponding dual space consists of all linear functionals on v, Here the dual space of normed spaces have been discussed.

6.2 OBJECTIVES

The Students will be able to,

- Witness the equivalent conditions on continuity and boundedness of linear operators
- Determine the boundedness of a linear operator in finite dimensional space.
- Dual space of normed space and suitable examples

6.3 BOUNDED OR CONTINUOUS LINEAR OPERATOR

Definition: 6.1.1A continuous linear operator is a linear transformation between two vector spaces and it is defined to be ,

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \ni : \|x - y\| < \delta \Rightarrow \|Ax - Ay\| < \epsilon$$

Then the operator A between these normed spaces is continuous.

Definition: 6.1.2 Let X and Y be normed spaces and T: D(T)→ Y a linear operator, where $D(T)$ is a subset of X. The operator T is said to be bounded if there is a real number C such that for all $x \in D(T)$,

$\|Tx\| \leq c\|x\|.$

Then T is said to be bounded linear operator.

Theorem: 6.1 (Continuity and boundedness). Let $T: D(T) \rightarrow Y$ be a linear operator, where D(T) is a subset of X,Y are normed spaces. Then

(i) T is bounded

(ii) T is continuous on D(T)

(iii)T is continuous at a single point x in D(T)

Proof: (i) implies (ii)

Assume that T is bounded.

case (i): If T=0, then clearly,

T is continuous on domain D(T).

Case(ii): If $T \neq 0$, then given $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{c}$, then for x,y in D(T) with $\|x - y\| < \delta.$

$$\|Tx - Ty\| = \|T(x - y)\|$$

$$\leq c\|x - y\|$$

$$\leq c.\delta$$

$$= c.\frac{\epsilon}{c}$$

$$= \epsilon$$

$$\|Tx - Ty\| < \epsilon$$

Therefore T is continuous on domain D(T).

(iii)implies (iii) is an obvious result from the statement.

(iv)implies (i)

Let $x_0 \in D(T)$ and T is continuous at x_0. Since T is continuous at x_0, given $\epsilon > 0, \exists\, \delta > 0, \ni y \in D(T)$ and $\|y - x_0\| < \delta.$

$\|Ty - Tx_0\| < \epsilon.$

Let $y \in D(T)$ *with* $y \neq 0$ *and choose,* $x = \frac{y}{\|y\|}\frac{\delta}{2} + x_0$.

$$\|x - x_0\| = \left\|\frac{y}{\|y\|}\frac{\delta}{2} + x_0 - x_0\right\|$$

$$= \left\|\frac{y}{\|y\|}\frac{\delta}{2}\right\|$$

$$= \frac{\delta}{2} < \delta$$

$$\Rightarrow \|x - x_0\| < \delta$$

Similarly, $\|Tx - Tx_0\| < \in$

$$\|T(x - x_0)\| = \left\|T\left(\frac{y}{\|y\|}\frac{\delta}{2} + x_0 - x_0\right)\right\|$$

$$= \left\|T\left(\frac{y}{\|y\|}\frac{\delta}{2}\right)\right\|$$

$$= \frac{\delta}{2\|y\|}\|Ty\|$$

$$\Rightarrow \|Ty\| < \frac{2\in}{\delta}\|y\|$$

Therefore, T is bounded.

Lemma: 6.2 Let T be a bounded linear operator defined as $\|Tx\| \leq c\|x\|$. Then,

(a) An alternative formula for norm of T is,

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$$

(b) The norm defined by $\|T\| = \sup_{\substack{x \in D(T) \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|}$ satisfies (N_1) to (N_4) of definition of norm

Proof: (a) Let $\|x\| = a$ and set $y = \left(\frac{1}{a}\right)x$, *where* $x \neq 0$.Then,

$$\|y\| = \|x\|/a = 1$$

Since T is linear,

$$\|T\| = sup_{\substack{x\in D(T)\\ \|x\|\neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$= sup_{\substack{x\in D(T)\\ \|x\|\neq 0}} \left\|T\left(\frac{1}{a}x\right)\right\|$$

$$= sup_{\substack{y\in D(T)\\ \|y\|=1}} \|Ty\|$$

$$Hence, \quad \|T\| = sup_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$$

(b)We know that the norm (N1) is obvious from the definition and

$\|0\| = 0, \quad from \quad \|T\| = 0$

We have $Tx = 0 \quad \forall\, x \in D(T), so\ that \quad T = 0$

Hence (N2) holds, (N3) is obtained from,

$$sup_{\|x\|=1}\|\alpha Tx\| = sup_{\|x\|=1}|\alpha|\|Tx\|$$

$$= |\alpha| sup_{\|x\|=1}\|Tx\|$$

Where, $x \in D(T)$, finally (N4) follows from

$$sup_{\|x\|=1}\|(T_1 + T_2)x\| = sup_{\|x\|=1}\|T_1x + T_2x\|$$

$$\leq sup_{\|x\|=1}\|T_1x\| + \|T_2x\|$$

Hence the proof.

Example: 6.1.3

Identity operator: The identity operator $I: X \to X$ on a normed space $x \neq 0$, is bounded and has norm $\|I\| = 1.$

Zero operator: The zero operator $0: X \to Y,$ on a normed space X is bounded and has norm $\|0\| = 0.$

Differentiation operator: Let X be the normed space of all polynomial on J=$[0,1]$ with norm given by

$\|x\| = max|x(t)|, t \in J.$

A differentiation operator T is defined on X by,

$Tx(t) = x'(t)$

Where the prime denotes the differentiation with respect to t. This operator is linear but not bounded.

Theorem:6.3 (Finite dimension) If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof: Let $\{e_1, e_2, \dots \dots e_n\}$ be the basis on domain D(T).

For every $x \in D(T)$, x can be written as,

$x = \sum_{i=1}^{n} \alpha_i e_i$ for some scalars α_i

$$\text{Now, } \|Tx\| = \|T \textstyle\sum_{i=1}^{n} \alpha_i e_i\|$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \|Te_i\|$$

$$\leq k \sum_{i=1}^{n} |\alpha_i|$$

$$\text{Where } k = max_{1 \leq i \leq n} \|Te_i\|$$

$$\text{Therefore, } \|Tx\| \leq \frac{k}{c} \|\textstyle\sum_{i=1}^{n} \alpha_i e_i\|$$

$$= \frac{k}{c} \|x\|$$

$$\|Tx\| \leq \frac{k}{c} \|x\|$$

Hence, T is bounded.

Thus, T is bounded.

6.4 DUAL SPACE

Definition: 6.1.4 Two normed spaces X and Y are said to be isomorphic if there exists a linear operator $T: X \rightarrow Y$ such that,

(i) $\|x\| = \|Tx\| \quad \forall\, x \in X$

(ii) T is on-to

Note: One to one follows from (i)

Definition: 6.1.5 Let X be a normed space, the dual space of X is defined by the normed space of all bounded linear functionals on X with point wise addition and point wise multiplication and the norm is defined as,

$$\|f\| = sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|}$$

The dual space of X is denoted by x'

Example: 6.1.6 The dual space of R^n is R^n.

Proof: Let $(\alpha_1, \alpha_2, \dots \alpha_n) \in R^n$

Define: $f_\alpha : R^n \to R$ by $f_\alpha(\beta_1, \beta_2, \dots, \beta_n) = \sum_{i=1}^{n} \alpha_i \beta_i$

Then f_α is clearly linear.

Since $R^n$is finite dimensional

$\Longrightarrow f_\alpha \epsilon (R^n)'$

Conversely, let $f \in (R^n)'$

Put, $\alpha_i = f(e_i) \quad \forall\, i = 1, 2, \dots n$ where (e_i) is the standard basis of R^n

Then for every $x \in R^n$,

$$x = \sum_{i=1}^{n} \delta_i f(e_i)$$

$$= \sum_{i=1}^{n} \delta_i \alpha_i$$

$$= f_\alpha(\gamma_1, \gamma_2, \dots, \gamma_n)$$

$$= f_\alpha(x) \;\; and \;\; \alpha \epsilon R^n$$

There is a one to one correspondence between R^n and $(R^n)'$

Further, $|f_\alpha(x)| = |\sum_{i=1}^{n} \alpha_i \beta_i|$

$$\leq \left(\sum_{i=1}^{n} \alpha_i^2\right)^{1/2} \left(\sum_{i=1}^{n} \beta_i^2\right)^{1/2}$$

By Schwartz inequality,

$\Longrightarrow \|f_\alpha\| \leq \|\alpha\|$

Now $\|f_\alpha\| = sup_{0\neq x \in R^n} \frac{|f_\alpha(x)|}{\|x\|} \geq \frac{|f_\alpha(x)|}{\|\alpha\|}$ where $\alpha \neq 0$

$\Rightarrow \|f_\alpha\| \geq \frac{\sum_{i=1}^{n} \alpha_i^2}{\|\alpha\|} = \|\alpha\|$

$\Rightarrow \|f_\alpha\| = \|\alpha\| \;\; if\ \alpha \neq 0$

If $\alpha = 0, then \;\; f_\alpha = 0 \Rightarrow \|\alpha\| = 0 = \|f_\alpha\|$

Therefore, $\|\alpha\| = \|f_\alpha\| \quad \forall\ \alpha \in R^n$.

Example: 6.1.7 The dual space of $l^1\ is\ l^\infty$, where, $l^1 = \{(x_n) < +\omega\}\ and$

$l^\infty with the norm \|x_n\|_1 = \sum_{n=1}^{\infty} |x_n|$

Proof: Let $x = (x_n) \in l^\infty$ be arbitrary,

Define: $f_x : l^1 \to R, by \;\; f_x(y_n) = \sum_{n=1}^{\infty} x_n y_n \;\; \forall \;\; (y_n) \in l^1$

Since, $|x_n y_n| \leq |x_n| \|y_n\|_\infty \forall\ n \in N$,

By comparison test,

$$\sum_{n=1}^{\infty} |x_n y_n| < +\infty$$

Therefore, $f_x : l^1 \to R$

Clearly $f_x\ is\ linear$.

Conversely, If $(\ l^1)^1$, then $f(e_n) = x_n \;\; \forall n \in N$

Where $\{e_n\}$ is the standard schauder basis in l^1

(i.e.,) $\qquad e_1 = (1,0,0, \ldots\ldots)$

$e_2 = (0,1,0,\ldots\ldots)$

$e_3 = (0,0,1,\ldots\ldots)$ and so on

$e_n = (0,0,0,\ldots.1,0.)$

If $y \in l^1, then\ \ y = (y_n)$ and $\sum_{n=1}^{\infty}|y_n| < +\infty$

$$\Rightarrow y = \sum_{n=1}^{\infty} y_n e_n$$

$$\Rightarrow f(y) = f\left(\sum_{n=1}^{N} y_n e_n\right)$$

$$= f\left(\lim_{n\to\infty} \sum_{n=1}^{N} y_n e_n\right)$$

$$= \lim_{n\to\infty} \sum_{n=1}^{N} y_n f(e_n)$$

$$= \sum_{n=1}^{\infty} y_n x_n$$

$$= f_x(y),\ where\ x = (x_n)$$

Claim: $(x_n) \in l^{\infty}$

For each $m \in N$, $\quad |f_x(e_m)| \leq \|f\|\|e_m\|$

$$\Rightarrow |x_m| \leq \|f\|\ \forall\ m \in N$$

$$\Rightarrow \|x\|_{\infty} = sup_{n\geq 1}|x_n| \leq \|f\| \leq \|f_x\|$$

$$\Rightarrow x \epsilon\ l^{\infty}$$

For $x \epsilon\ l^{\infty}, y \epsilon\ l^1$;

$$|f_x(y)| = \left|\sum_{n=1}^{\infty} x_n y_n\right|$$

$$\leq \sum_{n=1}^{\infty} |x_n y_n|$$

$$\leq \|x\| \sum_{n=1}^{\infty} |y_n|$$

$= \|x\|_{\infty}\|y\|_1$

$\Longrightarrow \|f\|_x \leq \|x\|_{\infty}$

$$\Longrightarrow \|f\|_x = \|x\|_{\infty}$$

Hence the proof.

6.5 EXERCISE

1. Show that inverse of a bounded linear operator need not to be bounded.
2. Show that an operator T is bounded iff it is continuous.
3. Show that the dual space of c_0 is l^1.
4. If X is a normed space and $\dim x = \infty$, show that the dual space of x' is not identical with the algebraically dual space of x^*

UNIT VII: INNER PRODUCT SPACE

Structure

7.1 INTRODUCTION

Here we will introduce inner product space and orthonormal sets and sequences and also gives some examples. We differentiate spaces which are inner product space or not. We will discuss about orthonormality of Hilbert spaces and some of its examples. Some important results in Hilbert spaces are determined by theorems.

7.2 OBJECTIVES

The students will be able to

- To understand the inner product space.
- Describe orthonormality of Hilbert space.
- Identify which spaces are inner product.
- Understand parallelogram law.
- To solve the problems related to inner product space.

7.3 INNER PRODUCT

Let X be a vector space over K, where K = $\mathbb{R}$ or $\mathbb{C}$. A function $\langle\cdot,\cdot\rangle$ from

$X \times X \to K$ is called an inner product (or a scalar product) on X if

(i) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(ii) $\langle x, y\rangle = \alpha\langle x, y\rangle$, $\forall \alpha \in K$, $\forall x, y \in X$

(iii) $\langle x, y\rangle = \langle \overline{y, x}\rangle$, $\forall x, y \in X$

(iv) $\langle x, x\rangle \geq 0$, $\forall x \in X$ and $\langle x, x\rangle = 0$ iff $x = 0$

Example:

1. $\mathbb{R}^n$ is an inner product space with respect to the inner product defined by
   $\langle (x_1, x_2, \ldots x_n), (y_1, y_2, \ldots y_n) \rangle = \sum_{k=1}^{n} x_k y_k$
2. $\mathbb{C}^n$ is an inner product space with respect to the inner product defined by
   $$\langle (x_1, x_2, \ldots x_n), (y_1, y_2, \ldots y_n) \rangle = \sum_{k=1}^{n} x_k \overline{w_t}$$
3. $l^2$is an inner product space with respect to the inner product defined by

$\langle (x_n), (y_n) \rangle = \sum_{k=1}^{\infty} x_n y_n$

{Here $l^2 = \{(x_n) : x_n \in \mathbb{R}\ \forall n \in \mathbb{N}, \left(\sum_{k=1}^{\infty} x_n^2\right)^{1/2} < \infty\}$}

4. $L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \begin{array}{c} f \text{ is Lebesgue measurable, } \int \\ |f(x)|^2 dx < \infty \end{array} \right\}$

is an inner product space with respect to the inner product defined by

$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx$

Results:

1. **Schwartz inequality:**

   If x,y $\in$ X, then $|\langle x, y \rangle| \leq \langle x, x \rangle \ \langle y, y \rangle$.

   Proof:

   Let $\alpha \in K$ be arbitrary.

$0 \leq \langle x - \alpha y, x - \alpha y \rangle = \langle x, x - \alpha y \rangle + \langle -\alpha y, x - \alpha y \rangle$

$= \langle x, x \rangle + \langle x, -\alpha y \rangle - \alpha(\langle y, x \rangle + \langle y, -\alpha y \rangle)$

$= \langle x, x \rangle - \bar{\alpha}\langle x, y \rangle - \alpha\langle y, x \rangle + \alpha\bar{\alpha}\langle y, y \rangle$

If $y = 0$, then clearly , $\langle x, y \rangle = 0$

$\langle x, 0 \rangle = \langle x, 0 + 0 \rangle = \langle x, 0 \rangle + \langle x, 0 \rangle \Rightarrow \langle x, 0 \rangle = 0$

If $y = 0$, by definition, $\langle y, y \rangle = 0$

$\therefore |\langle x,y\rangle|^2 = 0 = \langle x,x\rangle\langle y,y\rangle$

$y \neq 0$, $let \quad \alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$ then (1)

$$0 \leq \quad \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\langle y,y\rangle} - \frac{\langle x,y\rangle\langle y,x\rangle}{\langle y,y\rangle} + \frac{\langle x,y\rangle}{\langle y,y\rangle}\frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle y,y\rangle$$

$$\leq \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$$

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$$

2. Every inner product space X is a normed space with the norm defined by

$$\| x \| = \sqrt{\langle x,x\rangle}, \forall x \in X.$$

(i) clearly $\| x \| = \sqrt{\langle x,x\rangle} \geq 0, \forall x \in X$.

(ii) $\| x \| = 0 \Leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0$ (By condition 4)

(iii) $\| \alpha x \| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\bar{\alpha}\langle x,x\rangle}$

$$= \sqrt{|\alpha|^2\langle x,x\rangle}$$

$$= |\alpha| \; \| x \|$$

(iv) Let $x, y \in X$,

$$\| x + y \|^2 = \langle x+y, x+y\rangle$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= \| x \|^2 + \| y \|^2 + 2\mathrm{Re}\langle x,y\rangle (\because \overline{\langle x,y\rangle} = \langle y,x\rangle)$$

$$\leq \| x \|^2 + \| y \|^2 + 2|\mathrm{Re}\langle x,y\rangle|$$

$$\leq \| x \|^2 + \| y \|^2 + 2|\langle x,y\rangle|$$

$\leq \| x \|^2 + \| y \|^2 + 2 \| x \| \| y \|$ (By schwarhz inequality)

$$= (\| x \| + \| y \|)^2$$

$\therefore \| x + y \| \leq \| x \| + \| y \|$

Remark : The Schwarz inequality can be written as

$|\langle x, y \rangle| \leq \| x \| + \| y \|$

3. Paralogram Law:

In an inner product space X,

$$\| x + y \|^2 + \| x - y \|^2 = 2 (\| x \|^2 + \| y \|^2) \qquad \forall x, y \in X$$

$$\| x + y \|^2 + \| x - y \|^2 = \langle x + y , x - y \rangle + \langle x - y , x - y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle$$

$$- \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$= 2(\| x \|^2 + \| y \|^2)$$

7.4 ORTHOGONAL

Let X be an inner product space. If $x, y \in X$ and $\langle x, y \rangle = 0$. We say that x is Orthogonal to y (or) $x \perp y$

4. Pythagorian identity:

If $x, y \in X$ such that $x \perp y$, then $\| x + y \|^2 = \| x \|^2 + \| y \|^2$

Proof:

$$\| x + y \|^2 = \langle x + y, x + y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \| x \|^2 + \| y \|^2 \; (\because \langle x, y \rangle = \langle y, x \rangle = 0)$$

7.5 ORTHOGONAL SET

Let X be an inner product space. A set $S \subseteq X$ is said to be an orthogonal set if $\langle y, x \rangle = 0, \; \forall x, y \in S$ with $x \neq y$ and $x \neq 0 \forall x \in S$

1. If S is an orthogonal set in an inner product space X, then S is an independent set.

Proof:

Let $x_1, x_2, \ldots x_n \in S$ be arbitrary. If $\sum_{i=1}^{n} \alpha_i x_i = 0$ for some $\alpha_1, \alpha_2, \ldots \alpha_n \in K$, then$\langle \sum_{i=1}^{n} \alpha_i x_i, x_j \rangle = 0 \ \forall j \in \{1,2, \ldots n\}$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i \langle x_i, x_j \rangle = 0 \ \forall \ j \in \{1,2, \ldots n\}$$

$\alpha_j \langle x_j, x_j \rangle = 0 \ \ \forall j \in 1,2, \ldots n$

$\alpha_j \parallel x_j \parallel^2 = 0 \forall j \in 1,2, \ldots n$

$\alpha_j = 0, \forall j \in 1,2, \ldots n (\because \parallel x_j \parallel \neq 0)$

$\Rightarrow$ S is linearly independent.

2. Polarization Identity:

Let X be a complex inner product space. Then for every $x, y \in X$, we have

$\langle x, y \rangle = \frac{1}{4}(\parallel x + y \parallel^2 - \parallel x - y \parallel^2 + i(\parallel x + iy \parallel^2 - \parallel x - iy \parallel^2))$

Proof:

$\parallel x + y \parallel^2 - \parallel x - y \parallel^2 = \langle x + y, x + y \rangle - \langle x - y, x - y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle - [\langle x, x \rangle + \langle x, -y \rangle + \langle -y, x \rangle + \langle -y, -y \rangle$

$= 2(\langle x, y \rangle + \overline{\langle x, y \rangle})$

$i(\parallel x + iy \parallel^2 \parallel x - iy \parallel^2)) = \langle x + iy, x + iy \rangle - \langle x - iy, x - iy \rangle$

$= \langle x, x \rangle + \langle x, iy \rangle + \langle iy, x \rangle + \langle iy, iy \rangle - [\langle x, x \rangle + \langle x, -iy \rangle$

$+ \langle -iy, x \rangle + \langle -iy, -iy \rangle]$

$= 2(\langle x, iy \rangle + \langle iy, x \rangle)$

$= 2\langle x, y \rangle - 2\langle y, x \rangle$

$\parallel x + y \parallel^2 - \parallel x - y \parallel^2 + i(\parallel x + iy \parallel^2 \parallel x - iy \parallel^2)$

$= 2\langle x, y \rangle + 2\langle y, x \rangle + 2\langle x, y \rangle - 2\langle y, x \rangle$

$$= 4\langle x, y\rangle$$

$$\therefore \langle x, y\rangle = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2 + i(\| x + iy \|^2 - \| x - iy \|^2))$$

3. If $(X, \|.\|)$ is a normed space satisfying the parallogram law, then X is an inner product space.

4. Let X be an inner product space. If $x \in X \ni \langle x, u\rangle = \langle x, v\rangle \forall u, v$ then $u = v$

Proof:

Given that $\langle x, u\rangle = \langle x, v\rangle \forall x \in X$

$$\Rightarrow \langle x, u - v\rangle = 0 \ \ \forall x \in X$$

$$\Rightarrow \langle u - v, u - v\rangle = 0$$

$$\| u - v \|^2 = 0$$

$$\| u - v \| = 0$$

$$u - v = 0$$

$$u = v$$

5. Let X be an inner product space. Then inner product is continuous on X (or) prove that $\langle x_n, y_n\rangle \to \langle x, y\rangle \ as \ n \to \infty$ whenever $x_n \to x, y_n \to y \ as n \to \infty$.

Proof:

We have $\| x_n - x \| \to 0, \| y_n - x \| \to 0 \ as \ n \to \infty$

$$0 \le |\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x, y\rangle + \langle x_n, y\rangle - \langle x_n, y\rangle|$$

$$\le |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle|$$

$$\le |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\le \| x_n \| \| y_n - y \| + \| x_n - x \| \| y \|$$

$$\Rightarrow \| x \| .0 + 0. \| y \| \ as n \to \infty$$

$\Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$

6. If $x_n \perp y \ \forall n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ then $x \perp y$

Proof:

Now $\langle x, y\rangle = \langle \lim_{n\to\infty} x_n, y\rangle$

$= \lim_{n\to\infty} \langle x_n, y\rangle$

$= 0 (since\, x_n \perp y)$

$\Rightarrow x \perp y$

7. Prove that l^p is not an inner product space if $p \neq 2$.

We show that parallelogram law fails in l^p if $p \neq 2$.

Let (x_n) be $(1,1,0,0,\dots)$ and (y_n) be $(1,-1,0,0,\dots)$

$$\| (x_n) \|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} = (1+1)^{1/p} = 2^{1/p}$$

$$\| (x_n) + (y_n) \|_p = \| (2,0,0,\dots) \|_p = (2^p)^{1/p} = 2$$

$$\| (x_n) - (y_n) \|_p = \| (0,2,0,0,\dots) \|_p = (2^p)^{1/p} = 2$$

$$\| (x_n) + (y_n) \|_p^2 + \| (x_n) - (y_n) \|_p^2 = 8 \neq 2\left(2^{\frac{2}{p}} + 2^{\frac{2}{p}}\right)$$

$\therefore p \neq 2$

8. Prove that $C[a,b]$ is not a product space.

Proof:

We show tht parallelogram law is not satsified in $C[a,b]$.

Let $f(t) = 1 \forall t \in [a,b]$ and $g(t) = \frac{t-a}{b-a} \forall t \in [a,b], f, g \in C[a,b]$

$\| f \| = \sup_{t\in[a,b]} |f(t)| = 1, \| g \| = 1$

But $\| f + g \| = \sup_{t\in[a,b]} (1 + \frac{t-a}{b-a}) = 2$

$$\| f - g \| = \sup_{t\in[a,b]}(1 - \frac{t-a}{b-a}) = 1$$

$$\| f + g \|^2 + \| f - g \|^2 = 5 \neq 4 = 2(\| f \|^2 + \| g \|^2)$$

$\therefore C[a,b]$ is not an inner product space.

7.6 ORTHONORMAL

A subset M of a Hilbert space is said to be orthonormal

if $\langle x,y\rangle \begin{cases} 1 & if x = y \\ 0 & if x \neq y \end{cases}, \forall x,y \in H$

In particular, a sequence (x_n) is said to be orthonormal if

$\langle x_n, x_m\rangle = \delta_{m,n} \forall m,n \in \mathbb{N}$ where $\delta_{m,n}$ is the kronecker delta .

Example:

Let $L^2([0,2\pi])$ be the Hilbert space consisting of real valued function on $[0,2\pi]$ with the inner product $\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)dt, \forall f,g \in L^2([0,2\pi])$

If

$$f_0(t) = \frac{1}{\sqrt{2\pi}} \forall t \in [0,2\pi]$$

$$f_n(t) = \frac{cosnt}{\sqrt{\pi}} \forall t \in [0,2\pi]$$

$$g_n(t) = \frac{sinnt}{\sqrt{\pi}} \forall t \in [0,2\pi]$$

then $\{f_0, f_n, g_n : n \in \mathbb{N}\}$ is an orthonormal set in $L^2([0,2\pi])$

$$\| f_0 \|_2^2 = \langle f_0, f_0\rangle = \int_0^{2\pi} \frac{1}{2\pi} dt = 1$$

$$\langle f_n, f_n\rangle = \int_0^{2\pi} \frac{cosnt}{\pi} dt = 1$$

Theorem:7.4.1

Let (e_n) be an orthonormal sequence in a Hilbert space H. If $x \in H$, then $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \| x \|^2$

Proof:

First we prove that for any $m \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \| x \|^2$$

$$0 \leq \| x - \sum_{n=1}^{m} \langle x, e_n\rangle e_n \|^2$$

$$= \langle x - \sum_{n=1}^{m} \langle x, e_n\rangle e_n, x - \sum_{n=1}^{m} \langle x, e_n\rangle e_n\rangle$$

$$= \langle x, x\rangle - \sum_{n=1}^{m} \overline{\langle x, e_n\rangle}\langle x, e_n\rangle - \sum_{n=1}^{m} \langle x, e_n\rangle\langle e_n, x\rangle + \sum_{n=1}^{m} \sum_{j=1}^{m} \langle x, e_n\rangle\overline{\langle x, e_j\rangle}\langle e_n, e_j\rangle$$

$$= \| x \|^2 - 2\sum_{n=1}^{m} |\langle x, e_n\rangle|^2 + \sum_{n=1}^{m} |\langle x, e_n\rangle|^2$$

$$= \| x \|^2 - \sum_{n=1}^{m} |\langle x, e_n\rangle|^2$$

$$\Rightarrow \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \| x \|^2$$

Since this inequality is true for every $m \in \mathbb{N}$, allowing $m \to \infty$, we get

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \| x \|^2$$

Remark:

If (e_n) is an orthonormal sequence then (e_n) is linearly independent.

7.7 GRAM SCHMIDT'S PROCESS

To obtain an orthonomal sequence (e_n) from a sequence (x_n) of independent vectors such that span $\{x_1, x_2, \dots x_n\}$ = span$\{e_1, e_2, \dots e_n\} n \in \mathbb{N}$.

Step:1

Let $e_1 = \frac{x_1}{\|x_1\|}. Then \| e_1 \| = 1$

Step:2

$$e_2 = \frac{x_2 \quad \langle x_2, e_1\rangle e_1}{\| x_2 - \langle x_2, e_1\rangle e_1 \|} . Then \| e_2 \| = 1$$

and $\langle e_1, e_2\rangle = 0$

Step:3

Let $e_3 = \frac{x_3 - (\langle x_3, e_1\rangle e_1 + \langle x_3, e_2\rangle e_2)}{\|x_3 - (\langle x_3, e_1\rangle e_1 + \langle x_3, e_2\rangle e_2)\|} . Then \| e_2 \| = 1$

and $\langle e_1, e_3\rangle = 0$ simillarly $\langle e_2, e_3\rangle = 0$ Proceeding like this, at the n^{th}stage we get

$$e_n = \frac{x_2 - \sum_{k=1}^{n-1} \langle x_n, e_k\rangle e_k}{\| x_2 - \sum_{k=1}^{n-1} \langle x_n, e_k\rangle e_k \|}$$

clearly, $\langle e_n, e_k\rangle = 0 \forall k = 1,2, \ldots n-1, \| e_n \| = 1$

Theorem:7.5.1

If (e_α) be an orthonormal family of vectors in a Hilbert space H, for every $x \in H$ there exists atmost finite or countably infinitie number of Fourier co-efficients $\langle x, e_\alpha\rangle$ which are non-zero.

Proof:

For each $m \in \mathbb{N}$,

Let $S_m = \{\alpha : |\langle x, e_\alpha\rangle| > \frac{1}{m}\}$

we claim that, S_m is a finite set $\forall m \in \mathbb{N}$

Suppose S_m is not finite, then there exists a infinite sequence (e_α) of vectors from (e_α) belongs to S_m

By Bessel's inequality, we have

$$\sum_{n=1}^{\infty} |\langle x, e_\alpha\rangle^2| \leq \| x \|^2 \Rightarrow \sum_{n=1}^{\infty} |\langle x, e_\alpha\rangle|^2$$

is convergent, $|\langle x, e_\alpha\rangle|^2 \to 0 \; as \; n \to \infty$

$$\langle x, e_{\alpha}\rangle \to 0 as n \to \infty$$

For $\epsilon = 1/m$, then there exists $n \in \mathbb{N} \ni: |\langle x, e_{\alpha}\rangle| < \frac{1}{m}, \forall n \in \mathbb{N}$

which is a contradiction to $e_{\alpha_n} \in S_m \forall n \in \mathbb{N}$

$\therefore S_m$ is a finite set.

Let $S = \{\alpha: |\langle x, e_{\alpha}\rangle| \neq 0\}$

Claim: $S = \cup_{m=1}^{\infty} S_m$

clearly, $s_m \subseteq S \forall m \in \mathbb{N}$

Let $\alpha \in S$. Then $\langle x, e_{\alpha}\rangle \neq 0$.

$\Rightarrow |\langle x, e_{\alpha}\rangle| > 0$ then for $\epsilon = |\langle x, e_{\alpha}\rangle| > 0$

choose $n \in \mathbb{N} \ni: 1/n < |\langle x, e_{\alpha}\rangle|$

$\Rightarrow \alpha \in S_n \subseteq \cup_{m=1}^{\infty} s_m$

$\Rightarrow S = \cup_{m=1}^{\infty} S_m$

Since each S_m is finite, S is either finite or countably infinite.

Theorem:7.5.2

Let (e_n) be an orthogonal sequence in a Hilbert spae H and $x \in H$, then

(a) $\sum_{n=1}^{\infty} \alpha_n e_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges .

(b) If $\sum_{n=1}^{\infty} \alpha_n e_n$ converges to x, then $\alpha_n = \langle x, e_n\rangle \forall n \in \mathbb{N}$

(c) For every $x \in H$, $\sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$ converges.

Proof:

Let $S_n = \sum_{k=1}^{\infty} \alpha_k e_k$ and $\sigma_n = |\alpha_k|^2 \forall n \in \mathbb{N}$

for $m > n$, we have

$$\| S_m - S_n \|^2 = \| \sum_{k=n+1}^{m} \alpha_k e_k \|^2$$

$$= \langle \sum_{k=n+1}^{m} \alpha_k e_k, \sum_{k=n+1}^{m} \alpha_j e_j \rangle$$

$$= \sum_{k=n+1}^{m} \sum_{k=n+1}^{m} \alpha_k \bar{\alpha}_j \langle e_k, e_j \rangle$$

$$= \sum_{k=n+1}^{m} |\alpha_k|^2 = \sigma_m - \sigma_n$$

$\Rightarrow (S_n)$ is a cauchy in H if and only if (σ_n) if cauchy in K.

$\Rightarrow \sum_{n=1}^{\infty} \alpha_n e_n$ converges in $H \Leftrightarrow \sum_{n=1}^{\infty} |\alpha_n|^2$ converges in K

(b) $x = \sum_{n=1}^{\infty} \alpha_n e_n$ and $S_n = \sum_{k=1}^{n} \alpha_k e_k$

Fix $m \in \mathbb{N}$ be arbitrary ,

For $n > m$,

$$\langle s_n, e_m \rangle = \sum_{k=1}^{n} \alpha_k \langle e_k, e_m \rangle$$

$$= e_m$$

$$\alpha_m = \lim_{n \to \infty} \langle s_n, e_m \rangle = \langle \lim_{n \to \infty} s_n, e_m \rangle$$

$$= \langle x, e_m \rangle$$

(c) By Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \| x \|^2 < \infty$$

By using (a)$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ converges.

Theorem:7.5.3

Let (e_n) be an orthogonal sequence in a Hilbert space. If $x \in H$, then every rearrangement of $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ converges to the same limit.

Proof:

Let $m: \mathbb{N} \to \mathbb{N}$ be a bijection and $w_{m(n)} = e_n \forall n \in \mathbb{N}$.

Let $x_1 = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ and $x_2 = \sum_{m=1}^{\infty} \langle x, w_m \rangle w_m$

claim: $\langle x_1 - x_2, e_n \rangle = 0 \forall n \in \mathbb{N}$

$$\langle x_1 - x_2, e_n \rangle = \langle \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n - \sum_{m=1}^{\infty} \langle x, w_m \rangle w_m, e_p \rangle$$

$= \langle x, e_p \rangle - \langle x, w_{m(p)} \rangle$

$= \langle x, e_p \rangle - \langle x, e_p \rangle$

$= 0$

claim: $\langle x_1 - x_2, w_m \rangle = 0 \forall m \in \mathbb{N}$

$\langle x_1 - x_2, w_p \rangle = \langle \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n - \sum_{m=1}^{\infty} \langle x, w_m \rangle w_m, w_p \rangle$

$= \langle x, e_k \rangle - \langle x, w_p \rangle$

$= \langle x, e_p \rangle - \langle x, e_p \rangle$

$= 0$

Now $\| x_1 - x_2 \|^2 = \langle x_1 - x_2, x_1 - x_2 \rangle = 0$

$\Rightarrow \| x_1 - x_2 \| = 0 \Rightarrow x_1 = x_2$

7.8 TOTAL ORTHONORMAL SET

Let H be a Hilbert space. An orthonormal set M is said to be total if the span of M is dense in H.

Remark:

A total orthonormal set is called an orthonormal basis and M is a total orthonormal set if and only if M is orthonormal and $M_{\perp} = \{0\}$ (that is, if $x \perp m \; \forall m \in M$, then $x = 0$).

Theorem:7.6.1.

Let H be a Hilbert space and $M \subseteq H$ be an orthonormal set. M is total if and only if $\| x \|^2 = \sum_k |\langle x, e_k \rangle|^2$, where $(e_k) \subseteq M \ni: \langle x, e_k \rangle \neq 0 \forall x \in H$.

Proof:

Assume that $\| x \|^2 = \sum_k |\langle x, e_k \rangle|^2 \forall x \in H$ To prove M is total, (we show that $x \perp m \forall m \in H$)

$\Rightarrow x = 0$

By assumption, we have $\langle x, e_k \rangle = 0, \forall k$

$$\Rightarrow \sum_k |\langle x, e_k \rangle|^2 = 0 \Rightarrow \| x \| = 0 \Rightarrow x = 0$$

$\Rightarrow M$ is total

Conversly, assume that M is total. Let $y = \sum_k \langle x, e_k \rangle e_k$ (By theorem this series converges)

Claim: $x - y \perp m \forall m \in M$

case(i): $\langle x, m \rangle = 0 \Rightarrow m \neq e_k \forall K$

$$\langle x - y, m \rangle = \langle x, m \rangle - \langle y, m \rangle$$

$$= 0 - \sum_m \langle x, e_k \rangle \langle e_k, m \rangle$$

$$= 0$$

case(ii): $\langle x, m \rangle \neq 0 \Rightarrow m = e_k for some K$

$$\langle x - y, m \rangle = \langle x, e_k \rangle - \langle y, e_k \rangle$$

$$= \langle x, e_k \rangle - \sum_m \langle x, e_j \rangle \langle e_j, e_k \rangle$$

$$= \langle x, e_k \rangle - \langle x, e_k \rangle$$

$$= 0$$

$\Rightarrow x - y \perp m \forall m \in M \Rightarrow x - y = 0$ (since M is total)

$$\Rightarrow y = \sum_k \langle x, e_k \rangle e_k$$

Now, $\| x \|^2 = \langle x, x \rangle$

$$= \langle \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j \rangle$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \langle e_k, e_j \rangle$$

$$= \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

Hence the theorem follows.

7.9 EXERCISE

1. Prove that $\langle x, y\rangle = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2)$

2. Prove that Re $\langle x, y\rangle = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2)$

3. Prove that Im $\langle x, y\rangle = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2)$

3. If an inner product space X is real , show that the condition $\| x \| = \| y \|$ implies $\langle x + y, x - y\rangle = 0$. What does this mean geomentrically if X=R^2?

UNIT VIII: ANNIHILATORS AND PROJECTIONS

Structure

8.1 INTRODUCTION

This unit discuss about the annihilators and projections in Hilbert spaces. Also discuss about the important results of annihilators and theorems. We will provide some examples to projection map on Hilbert spaces. At the end, we provide exercises to readers.

8.2 OBJECTIVES

To describe the annihilator set in Hilbert space.

To define the projection maps on Hilbert space.

Describe the orthogonal projection maps on Hilbert space.

8.3 ANNIHILATOR

The annihilator $M^{\perp}$ of a set $M \neq \emptyset$ in an inner product space X is the set

$$M^{\perp} = \{x \in X \mid x \perp M\}.$$

Thus, $x \in M^{\perp}$ if and only if $\langle x, v\rangle = 0$ for all $v \in M$.at

Result:

Let $\phi \neq M \subseteq H$. Then

(i) $M^{\perp}$ is a subspace of H.

(ii) $M^{\perp}$ is closed in H

(iii) $M \subseteq M^{\perp\perp}$

(iv) If $A \subseteq B$, then $A^{\perp} \supseteq B^{\perp}$

Proof:

(i) Let $x_1, x_2 \in M^{\perp}, \propto, \beta \in k$. Then to prove $\alpha x_1 + \beta x_2 \in M^{\perp}$, we show that let $y \in M$ be arbitrary. Then $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$=0 $(since\ x_1 \in M^{\perp}, x_2 \in M^{\perp})$

(ii) Let $x \in M^{\perp}$, then there xists (x_n) in $M^{\perp}$ such that $x_n \to x$ as $n \to \infty$

For every $y \in M$, $\langle x, y\rangle = \langle \lim_{n\to\infty} x_n, y\rangle$

$$= \lim_{n\to\infty} \langle x_n, y\rangle$$

$$= 0 (since\ x_n \in M^{\perp} \forall n \in N)$$

$$\Rightarrow x \in M^{\perp}$$

$$\Rightarrow M^{\perp} is closed.$$

(iii) Since for every $x \in M$, $x \perp y \ \ \forall y \in M^{\perp}$ obviously $M \subseteq M^{\perp\perp}$.

(iv) Let $x \in B^{\perp} \Rightarrow x \perp y \, \forall y \in B$

$$\Rightarrow x \perp y \ \ \forall y \in A (since A \subseteq B)$$

$$\Rightarrow x \in A^{\perp}$$

$$\Rightarrow B^{\perp} \subseteq A^{\perp}$$

Lemma: 8.3.1

Let $\phi \neq M \subseteq H$. $M = M^{\perp\perp}$ if and only if M is a closed subspace of H.

Proof:

Obviously, $M = M^{\perp\perp}$.

$\Rightarrow M$ is a closed subspace of H.

Conversely assume that, M is a closed subspace of H.

WKT, $M \subseteq M^{\perp\perp}$

Now, Let $x \in M^{\perp\perp} \subset H$.

by previous theorem, $H = M \oplus M^{\perp}$

$\Rightarrow x = y + z, where\ y \in M, z \in M^{\perp}$

claim: $z \in M^{\perp\perp}$

Let $v \in M^{\perp}$, then $\langle z, v\rangle = \langle x - y, v\rangle$

$$= \langle x, v\rangle - \langle y, v\rangle = 0$$

$$\Rightarrow z \perp M^{\perp} \Rightarrow \langle z, z\rangle = 0 \Rightarrow z = 0 \Rightarrow x = y \in M.$$

Lemma: 8.3.2

Let M be a non-empty subset of a Hilbert space H. Then span of M is dense in H if and only if $M^{\perp} = \{0\}$.

Proof:

Let $v = span\ of\ M$

Assume that V is dense in H.

Let $x \in M^{\perp} \subseteq H$

since $\bar{V} = H$, there exists (x_n) from $V \ni: x_n \to x\ as\ n \to \infty$.

Since $x \in M^{\perp}$, we have $x \perp V \Rightarrow x \perp x_n \forall\, n \in \mathbb{N}$

$$\Rightarrow \langle x, x_n\rangle = 0 \quad \forall n \in \mathbb{N}$$

Now, $\langle x, x\rangle = \langle \lim_{n\to\infty} x_n, x\rangle = \lim_{n\to\infty}\langle x_n, x\rangle = 0$

$$\Rightarrow x = 0$$

$$\Rightarrow M^{\perp} = \{0\}$$

Conversely, assume that $M^{\perp} = \{0\}$

clearly, $\bar{V}$ is a closed subspace of H.

$\Rightarrow H = \bar{V} \oplus \bar{V}^{\perp}$ (by theorem)

Let $z \in \bar{V}^{\perp} \Rightarrow z \perp y \ \ \forall\, y \in \bar{V} \Rightarrow z \perp y \ \forall y \in M$

$\Rightarrow x \in M^{\perp} \Rightarrow x = 0$

$$\Rightarrow \bar{V}^{\perp} = \{0\}$$

$\Rightarrow H = \bar{V} \Rightarrow v$ is dense in H..

8.4. ORTHOGONAL PROJECTION

The linear operator $P_V : H \rightarrow H$ that maps x to v is called the orthogonal projection onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all bounded self-adjoint operators P such that $P^2 = P$.

Examples:

(i) Let Y be a orthogonal subspace of H then the map f is from H onto Y.

(ii) The map g is from Y onto Y.

(iii) The map H is from $Y^{\perp}$ *onto* $\{0\}$

8.5 EXERCISE

1. Let A and B $\supset A$ be nonempty subsets of an inner product space X. Show that (a) A $\subset A^{\perp\perp}$ (b) $B^{\perp} \subset A^{\perp}$ (c) $A^{\perp\perp\perp} = A^{\perp}$

2. Show that the annihilator $M^{\perp}$ of a set $M \neq \phi$ is an inner product space is a closed subspace of X.

3. Show that a subspace Y of a Hilbert space H is closed in H is closed in H if and only if Y$= Y^{\perp\perp}$.

4. If $M \neq \phi$ is any subset of a Hilbert space H, show that $M^{\perp\perp}$ is the smallest closed subspace of H which contains M, that is, $M^{\perp\perp}$ is contained in any closed subspace Y$\subset$ H such that Y$\supset M$.

UNIT IX: HILBERT SPACE

Structure

9.1 INTRODUCTION

In this unit, we introduce the concept of Hilbert spaces and convexity. It introduce the notion of complete space of Hilbert space through definitions and theorems. Finally this unit ends with some exercises.

9.2 OBJECTIVES

The students will be able to

Understand more about Hilbert space.

Describe the direct sum between two vector spaces.

Determine the isomorphism and Hilbert dimension.

9.3 HILBERT SPACE

A complete inner product space is called a Hilbert space.

Example:

1. Consider the space $\mathbb{R}^n$. For x = $(x_1, x_2, \ldots x_n)$ y = $(y_1, y_2, \ldots y_n)$, we define

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

This defines as inner product and the norm associated to it is the norm $\|\cdot\|_2$.

Thus l_2^n is a Hilbert space. In the case of $\mathbb{C}^n$, the inner product is given by

$$(x,y) = \sum_{i=1}^{n} x_i \bar{y}_i.$$

Again the norm $\| \cdot \|_2$.

9.4 CONVEX SET

A subset A of a vector space X is said to be a convex set if $(1-t)x + ty \in A\ \forall\, x, y \in A$ and $\forall\, t \in [0,1]$.

Theorem. 9.4.1

Let H be a hilbert space and M be a proper complete subspace of H. For every $x \in H/M$, there exists unique $y \in M \ni$:

$$\| x - y \| = \inf\{\| x - m \| : m \in M\}$$

Proof:

Let $\delta = \inf\{\| x - m \| : m \in M\}$

Then there exists a sequence (m_n) from $M \ni$: $\| x - m_n \| = \delta_n \forall n \in \mathbb{N}$ and $\delta_n \to \delta$ as $n \to \infty$ Let $V_n = x - m_n\ \forall\, n \in \mathbb{N}$

We claim that (m_n) is a cauchy sequence in M.

Now

$$\| m_i - m_j \|^2 = \| v_i - v_j \|^2 = -\| v_i + v_j \|^2 + 2(\| v_i \|^2 + \| v_j \|^2)$$

$$= -\| 2x - m_i + m_j \|^2 + 2(\delta_i^2 + \delta_j^2)$$

$$= 4 \| x - 1/2(m_i + m_j) \|^2 + 2(\delta_i^2 + \delta_j^2)$$

$$\leq -4\delta^2 + 2\delta_i^2 + 2\delta_j^2 \to 0 as i,j \to \infty.$$

$\Rightarrow (m_n)$ is a cauchy sequence in M. Since M is complete , $m_n \to y$ as $n \to \infty$ for some y$\in M$.

$$\therefore \| y - x \| = \| \lim_{n\to\infty} m_n - x \|$$

$$= \lim_{n\to\infty} \| m_n - x \|$$

$$= \lim_{n\to\infty} \delta_n = \delta$$

$$= \inf \| x - m \| : m \in M$$

To prove the uniqueness of y, let there is $y' \in M \ni$: $\| y' - x \| = \| y - x \| = \delta$

Now,

$$\| y - y' \|^2 = \| y - x - (y' - x) \|^2$$

$$= 2(\| y - x \|^2 + \| y' - x \|^2) - \| y - x + y' - x \|^2$$

$$= 4\delta^2 - \| y + y' - 2x \|^2$$

$$= 4\delta^2 - 4 \| 1/2(y + y') - x \|^2$$

$$\leq 4\delta^2 - 4\delta^2 = 0$$

$$\Rightarrow y = y'$$

Theorem: 9.4.2

Let M be a proper complete subspace of a Hilbert space H. If $x \in H/M$ and $y \in M \ni$: $\| x - y \| = \inf\{\| x - m \| : m \in M\}$ then $x - y \perp M$ $[(ie) x - y \perp m, \forall m \in M]$

Proof:

Suppose there exists $m_1 \in M \ni : \langle m_1, z \rangle \neq 0$ where $z = x - y$. For every $\alpha \in k$, Consider

$$\| z - \alpha m_1 \|^2 = \langle z - \alpha m_1, z - \alpha m_1 \rangle$$

$$= \langle z, z \rangle - \bar{\alpha}\langle z, m_1 \rangle - \alpha\langle m_1, z \rangle + |\alpha|^2 \langle m_1, m_1 \rangle$$

$$= \| z \|^2 - -\bar{\alpha}\langle z, m_1 \rangle - \alpha\langle m_1, z \rangle + |\alpha|^2 \, \| m_1 \|^2$$

Put $\alpha = \frac{\langle z, m_1 \rangle}{\|m_1\|^2}$ (Note that $m_1 \neq 0$) $\| z - \alpha m_1 \|^2 = \| z \|^2 - \frac{\overline{\langle z, m_1 \rangle}}{\|m_1\|^2} \langle z, m_1 \rangle$

$$\Rightarrow \| z - \alpha m_1 \|^2 = \| z \|^2 = \delta^2 - \frac{|\langle z, m_1 \rangle|^2}{\| m_1 \|^2} < \delta$$

Now $z - \alpha m_1 = x - y - \alpha m_1 = x - (y + \alpha m_1)$

$$\Rightarrow \| z - \alpha m_1 \| \geq \delta \,(since \;\; y + \alpha m_1 \in M)$$

which is a contradiction.

$\therefore \;\; z \perp M$.

From the above two results we have the following. Given a closed subspace Y of H and $x \in X$, there exists unique

$y \in Y \ni: \| y - x \| = \inf\{\| v - x \| : v \in Y\}$ and $x - y \perp Y$.

9.5 DIRECT SUM

Let X be a vector space. We say that X is a direct sum of its subspace A and B if for every $x \in X$ can be uniquely written as $x = y + z$, where $y \in A$ and $z \in B$. We write this by $X = A \oplus B$.

9.6 ORTHOGONAL

Let H be a Hilbert space and M be a non-empty subset of H. The orthogonal complement of M is defined by

$M^{\perp} = \{x \in H : x \perp y \;\; \forall \;\; y \in M\}$

Theorem: 9.5.1

Let Y be a closed subsapce of a Hilbert space H. Then $H = Y \oplus Y^{\perp}$

Proof:

By earlier results, for every $x \in H$, there exists unique $y \in Y \ni:$

$\| y - x \| = \inf\{\| v - x \| : v \in Y\} and x - y \perp Y$

Put$z = x - y$, then $x = y + z$ and $z \perp Y$ (or) $z \in Y^{\perp}$

$\Rightarrow H = Y + Y^{\perp}$

To prove $H = Y \oplus Y^{\perp}$, if $x = y + z = y' + z'$

such that $y, y' \in Y$ and$z, z' \in Y^{\perp}$ Then

$y - y' = z' - z \in Y \cap Y^{\perp}$

$\Rightarrow \langle y - y', y - y' \rangle = 0 \Rightarrow y - y' = 0 \Rightarrow y = y'$

$\Rightarrow z = z'$

Recall:

A topological space X is said to separable if X has a countable dense set.

Theorem: 9.5.2

Let H be a Hilbert space. Then

(a) H is countable.

(b) If H has a orthonomal basis, then H is separable.

Proof:

(a) Let A be a countable dense set in H. If B is an orthonormal basis of H, then we show that B is countable. If $x, y \in B$ with $x \neq y$, then

$$\| x - y \|^2 = \langle x - y, x - y \rangle$$

$$= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$= 2$$

$$\| x - y \| = \sqrt{2}$$

If $N_x - \{z \in H : \| z - x \| < \frac{\sqrt{2}}{3}\}$ and $N_y - \{z \in H : \| z - y \| < \frac{\sqrt{2}}{3}\}$, then since A is dense in H, for every $x \in B$, there exists at least one point say $z_x \in N_x \cap A$. Fix such $z_x \in N_x \cap A$,

Define $f : B \to A$ by $f(x) = z_x \forall x \in B f$ is 1-1

$\therefore$cordinality of $B \leq$ cordinality of A

sine A is countable, we get B is countable.

To prove let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of H.

Let $S = \{\sum_{k=1}^{n} q_k e_k : q_k \in Q(\ or \ Q \times Q) \ \forall \ k = 1, 2, \ldots n, n \in \mathbb{N}\}$

Let $\epsilon > 0$ be given. Since $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of H, for every $x \in H$, we have $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$

$$= \lim_{n \to \infty} \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$$

For this $\epsilon > 0$, there exists $N \in \mathbb{N} \ni: \| \sum_{k=N+1}^{\infty} \langle x, e_n \rangle e_n \| < \frac{\epsilon}{2}$

Since Q is dense in $\mathbb{R}$ for each k=1,2,...N

choose $q_k \in Q \ni |\langle x, e_k \rangle - q_k| < \frac{\epsilon}{2\sqrt{n}}$

If $\bar{g} y = \sum_{k=1}^{N} q_k e_k$, then $y \in S$

Claim: $y \in \{z \in H : \| z - x \| < \epsilon \}$

$$\| y - x \| = \| \sum_{k=1}^{N} q_k e_k - \sum_{k=1}^{N} \langle x, e_k \rangle e_k - \sum_{k=N+1}^{\infty} \langle x, e_n \rangle e_n \|$$

$$\leq \| \sum_{k=1}^{N} (q_k - \langle x, e_k \rangle) e_k \| + \| \sum_{k=N+1}^{\infty} \langle x, e_n \rangle e_n \|$$

$$< \sqrt{\sum_{k=1}^{N} | q_k - \langle x, e_k \rangle |^2} + \frac{\epsilon}{2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon$$

since S is countable, H is seperable.

9.7 ISOMORPHIC

Let H and $\tilde{H}$ be Hilbert spaces one a same field K. H is said to be isomorphic $\tilde{H}$ if there exists a linear bijection T from onto $\tilde{H}$ such that $\langle x, y \rangle = \langle T_x, T_y \rangle \forall x, y \in H.$

Example:

If $\{e_k\}_{k \in B}$ is an orthonormal basis of H, then the map $\Phi : H \to l^2(B)$ defined by $\Phi(x) = \langle x, e_k \rangle_{k \in B}$ is an isometric isomorphism of Hilbert spaces: it is a bijective linear mapping such that

$$\langle \Phi(x), \Phi(y) \rangle = \langle x, y \rangle$$

for all x, $y \in H$. The cardinal number of B is the Hilbert dimension of H. Thus every Hilbert space is isometrically isomorphic to a sequence space $l^2(B)$ for some set B.

9.8 HILBERT DIMENSION

Let H be a Hilbert space. The Hilbert dimension (or orthogonal dimension) of H is defined as the cordinality of any orthonormal basis of H.

Example:

$l^2(B)$ has an orthonormal basis indexed by B, its Hilbert dimension is the cardinality of B (which may be a finite integer, or a countable or uncountable cardinal number).

9.9 EXERCISE

1. Show that an inner product space of finite dimension n has a basis $\{b_1, b_2, \ldots b_n\}$ of orthonormal vectors.
2. If $x \perp y$ in an inner product space X, show that

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$

3. If an inner product space, $\langle x, u \rangle = \langle x, v \rangle$ for all x, show that u=v

UNIT X: REFLEXIVITY OF HILBERT SPACES

Structure

10.1 INTRODUCTION

The concept of Hilbert space named after David Hilbert, generalizes the notion of Euclidean space. It extends the method of vector algebra and calculus from the two dimensional Euclidean plane and three dimensional Euclidean space in to infinite dimensional space. In this section we discuss the reflexivity of Hilbert spaces.

10.2 OBJECTIVES

The Students will be able to,

Witness the detailed proof of reflexivity of Hilbert space.

Determine the results oriented to Separability.

Identify there flexivity of finite dimensional spaces.

10.3 REFLEXIVITY OF HILBERT SPACES

Definition: 10.1.1 (Reflexivity) A normed space X is said to be reflexive if,

$$R(C) = X''$$

Where $C: X \to X''$ is a canonical mapping.

Definition: 10.1.2 (Embeddable) X is said to be embeddable in a normed space Z. if X is isomorphic with a subspace of Z.

Lemma: 10.1 (Canonical mapping) The canonical mapping $C: X \to X''$ is an isomorphism of the normed space X on to the normed space R(C), the range of C.

Proof: Linearity of C as follows,

$C: X \to X''$ is linear.

Let $x_1, x_2 \in X$

Claim: $C(x_1 + x_2) = C(x_1) + C(x_2)$

$f_{x_1+x_2} = f_{x_1} + f_{x_2}$ on X'

Let $g \in X'$ be arbitrary. Then,

$$f_{x_1+x_2}(g) = (x_1 + x_2)g$$

$$= g(x_1) + g(x_2)$$

$$= f_{x_1}(g) + f_{x_2}(g)$$

$$= f_{x_1} + f_{x_2}(g)$$

Claim: For $\alpha \in K$ and $x \in X$, $C(\alpha x) = \alpha(Cx)$

Let $g \in X'$ be arbitrary

$$f_{\alpha x}(g) = (g)(\alpha x)$$

$$= \alpha(g(x))$$

$$= \alpha f_x(g)$$

Hence C is linear.

Claim: Bijective of C

For every fixed x in a normed space X, the functional g_x is a bounded linear functional on X', so that $g_x \in X''$, and has the norm

$\|g_x\| = \|x\|$

Hence we obtain, $\|g_x - g_y\| = \|g_{x-y}\| = \|x - y\|$

This shows that, C is isometric$\Rightarrow$ Injectivity.

Indeed if, $x \neq y$, $g_x \neq g_y$

Hence C is bijective implies one-one and on-to.

On-to implies R(C)=X

Hence the proof.

Theorem : 10.2 If a normed space X is reflexive, it is complete

(Hence a banach space).

Proof: Let $X'' \text{ is the}$ dual space of X' and X'' is complete

(By theorem: The dual space X' of a normed space X is a banach space)

Claim: X is complete

Reflexivity of X implies , $R(C) = X''$

Completeness of X now follows from that of X''(from lemma:10.2)

Theorem: 10.3 (Finite dimension) Every finite dimensional normed space is reflexive.

Proof: $l^p \text{ with } 1 < p < +\infty$ is reflexive.

Similarly, $L^p[a,b]$ with $1 < p < +\infty$ is reflexive.

It can also be proved that non-reflexive spaces are C[a, b], l^∞, and the subspaces c and c_0 of l^∞,

where C - Space of all convergent sequences of scalars

c_0 - The space of all sequences of scalars converging to zero.

Theorem: 10.4 Every Hilbert space is reflexive.

Proof: Let H be a Hilbert space and A:$H' \to H$ be the operator such that A(f)=Z where $f(x) = \langle x, z\rangle \forall\, x \in H \quad and \quad \|z\| = \|f\|$

(from Riesz's theorem) we define $\langle .,. \rangle' : H' \to k$ by,

$\langle f_1, f_2\rangle'$=$\langle Af_1, Af_2\rangle \quad \forall\, f_1, f_2 \in H'$

We know that $A(f_1 + f_2) = Af_1 + Af_2 \;\; \forall\, f_1, f_2 \in H'$

And $A(\alpha f) = \alpha(Af) \;\; \forall\, f_1, f_2 \in H', \alpha \in K$

Claim: $\langle .,. \rangle'$ is an inner product of H'

(i) $\langle f_1 + f_2, f_3 \rangle' = \langle Af_3 + A(f_1 + f_2) \rangle$

$= \langle Af_3 + Af_1 + Af_2 \rangle$

$= \langle Af_3, Af_1 \rangle + \langle Af_3, Af_2 \rangle$

$= \langle f_1, f_3 \rangle' + \langle f_2, f_3 \rangle'$

(ii) $\langle \alpha f_1, f_2 \rangle' = \langle Af_2, A(\alpha f_1) \rangle$

$= \langle Af_2, \alpha A(f_1) \rangle$

$= \alpha \langle Af_2, A(f_1) \rangle$

$= \alpha \langle f_1, f_2 \rangle'$

(iii) $\langle f_1, f_2 \rangle' = \langle Af_2, A(f_1) \rangle$

$= \overline{\langle Af_1, A(f_2) \rangle}$

$= \overline{\langle f_2, f_1 \rangle}$

(iv) $\langle f, f \rangle' = \langle Af, Af \rangle$

$= \|Af\|^2 \geq 0$

Therefore, H' is an inner product space with respect to $\langle .,. \rangle'$ and hence it is a Hilbert space.

To prove: H is a reflexive space, we show that the canonical map $c: H \rightarrow H''$ is on-to.

Let $h \in H'$

Then, by Reisz's theorem, $\exists\ f_0 \in H' \ni h(f) = \langle f, f_0 \rangle'\ \forall\ f \in H'$

Let A$f_0 = x$, we claim that c(x)=h (i.e.,) $g_x = h$

Now, $g_x(f) = f(x) = \langle x, z \rangle$ if $Af = Z$

$= \langle Af_0, Af \rangle$

$= \langle f, f_0 \rangle'$

= h(f)

Therefore, $g_x = h$

Thus, H is a reflexive space.

Result: 10.1.3 A separable normed space X with a non separable dual space X' cannot be reflexive.

Proof: l' is separable by the known result,

(i.e.,) The space $l^p \quad with\ 1 \leq p < +\infty$ is separable.

The dual space of $(l')' = l^\infty$

Where l^∞ is not separable

This implies that l' is non-reflexive space.

Lemma:10.5 (Existence of a functional) Let Y be a proper closed subspace of a normed space X. Let $x_0 \in X - Y$ be arbitrary and

$\delta = inf_{y \epsilon y} \|y - x_0\|$

The distance from $x_0 \quad to \quad Y$. Then there exists an $f \in \ x'$ such that

$\|f\| = 1, \quad f(y) = 0$ for all $y \in Y$, $\quad f(x_0) = \delta$

Proof: Let us consider the subspace Z is improper subspace of X spanned by Y and x_0, define on Z a bounded linear functional f by,

$f(z) = f(y + ax_0) = a\delta, y \in Y$

Every $z \in Z$ = span ($y \cup \{x_0\}$) has unique representation

$z = y + ax_0, \quad y \in Y$

Clearly, the linearity of f can be seen from the above argument.

Also, since Y is closed,

$\delta > 0, so\ that\ f \neq 0. Now\ \alpha = 0\ gives\ \ f(y) = 0\ \ for\ all\ y \in Y$

For $\alpha = 1$ and y=0, we have $f(x_0) = \delta$

We show that f is bounded. $\alpha = 0$ gives f(z)=0.

Let $\alpha \neq 0$ and $-(1/\alpha)y \in Y$, we obtain

$$|f(z)| = |\alpha|\delta = |\alpha| inf_{\bar{y}\in Y}\|\bar{y} - x_0\|$$

$$\leq |\alpha| \left\| -\frac{1}{\alpha}y - x_0 \right\|$$

$$= \|y + \alpha x_0\|$$

That is, $|f(z)| \leq \|z\|$.

Hence f is bounded and $\|f\| \leq 1$.

Claim: $\|f\| \geq 1$. By the definition of an infimum, Y contains a sequence (y_n) such that $\|y_n - x_0\| \to \delta$. let $z_n = y_n - x_0$. Then we have, $f(z_n) = -\delta$ with $\alpha = -1$.

Also, $\|f\| = sup_{\substack{z\in Z \\ z\neq 0}} \frac{|f(z)|}{\|z\|}$

$$\geq \frac{|f(z_n)|}{\|z_n\|}$$

$$= \frac{\delta}{\|z_n\|} \to \frac{\delta}{\delta} = 1$$

Hence $\|f\| \geq 1$

$so\ that\ \|f\| = 1$.

Theorem: 10.6 (Separability) If the dual space x' of a normed space x is separable, then x itself is separable.

Proof: We assume that x' is separable. Then the unit sphere,

$$U' = \{f \ \backslash \|f\| = 1\} subset of\ x'$$

Also contains a countable dense subset, say (f_n). Since $f_n \in U'$, we have

$$\|f_n\| = sup_{\|x\|=1}|f_n(x)| = 1$$

By the definition of supremum we can find points $x_n \in X$ of norm 1 such that,

$$|f_n(x_n)| \geq \frac{1}{2}$$

Let Y be the closure of span (x_n). Then y is separable because Y has s countable dense subset, namely, the set of all linear combinations of the x_n with coeffecients whose real and imaginary parts are rational.

We show that Y=X. suppose $y \neq x$. Then, since y is closed, by the above lemma there exists an $f \in x'$ with,

$$\|f\| = 1, \qquad f(y) = 0 \text{ for all } y \in Y, \qquad f(x_0) = \delta$$

For all y in Y. since, $x_n \in y$, we have $f(x_n) = 0$ and fro all n,

$$\frac{1}{2} \leq |f_n(x_n)| - |f_n(x_n) - f(x_n)|$$

$$= |(f_n - f)(x_n)|$$

$$\leq \|f_n - f\| \|x_n\|$$

Where, $\|x_n\| = 1$.

Hence, $\|f_n - f\| \geq \frac{1}{2}$,

But this contradicts the assumption that (f_n) is dense in U' because f is itself in U' and also $\|f\| = 1$.

Hence the proof.

10.4 EXERCISE

1. If a normed space X is reflexive, show that X' is reflexive.
2. Show that a Banach space X is reflexive iff its dual space X' is reflexive.
3. Prove that if f is a linear functional on the Hilbert space X, with null space N, then f is continuous iff N is a closed subspace.
4. Prove that a closed subspace of a reflexive Banach space is reflexive.

UNIT XI: RIESZ'S THEOREM

Structure

11.1 INTRODUCTION

In this unit we will introduce the basic concepts such as sesquilinear form and adjoint operators and also highlights the basic properties of adjoint operators. We will discuss the most important theorems of this unit is Riesz's theorem and Riesz's representation theorem. Some useful results for adjoint operator will be discussed and then we classify the bounded linear operator are described.

11.2 OBJECTIVES

Students will able to

To understand more about sesquilinear form and adjoint operator.

Describe the basic properties of adjoint operator.

Determine the classification of bounded linear operator.

To solve related results.

11.3 RIESZ'S THEOREM

Theorem: 11.3.1

Let H be a Hilbert space. For every bounded linear functional f on H, there exists unique $z \in H$ such that $f(x) = \langle x, z \rangle \ \forall x \in H$ and $\|f\| = \|z\|$.

Proof:

Case (1): $f = 0$, then obviously, we choose $z = 0$.

Case (2): $f \neq 0$

Then $H \neq 0$ and the null space $N(f)$ is a closed subspace of H.

By a theorem, we have $H = N(f) + N(f)^{\perp}$

$\therefore N(f)^{\perp} \neq 0$ (otherwise $H = N(f) \Rightarrow f = 0 \Rightarrow \Leftarrow$)

Choose $z_0 \in N(f)^{\perp}$ such that $z_0 \neq 0$

Let $V = f(x)z_0 - f(z_0)x$; for a given $x \in H$

Claim: $V \in N(f)$

$f(V) = f(f(x)z_0 - f(z_0)x)$

$\quad = f(x)f(z_0) - f(z_0)f(x)$

$\quad = 0$

$\Rightarrow V \in N(f) \Rightarrow \langle V, z_0 \rangle = 0$

Now, $0 = \langle V, z_0 \rangle = \langle f(x)z_0 - f(z_0)x, z_0 \rangle$

$\qquad = f(x)\langle z_0, z_0 \rangle - f(z_0)\langle x, z_0 \rangle$

$\Rightarrow f(x) = \frac{f(z_0)\langle x, z_0 \rangle}{\|z_0\|^2} = \langle x, \frac{\overline{f(z_0)}}{\|z_0\|^2} z_0 \rangle$

We take $z = \frac{\overline{f(z_0)}}{\|z_0\|^2} z_0$, then $z \in H$ and $f(x) = \langle x, z \rangle \ \forall x \in H$.

To prove uniqueness of this z,

Let $z_1, z_2 \in H \ \ni: f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle \forall x \in H$

$\Rightarrow \langle x, z_1 - z_2 \rangle = 0 \ \ \forall x \in H.$

$\Rightarrow$ In particular, for $x = z_1 - z_2$, we get

$\langle z_1 - z_2, z_1 - z_2 \rangle = 0 \Rightarrow \|z_1 - z_2\| = 0 \Rightarrow z_1 = z_2$

Since f is a bounded linear function

$$\|f\| = \sup_{\substack{x \in H \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \geq \frac{|f(z)|}{\|z\|} = \frac{|\langle z,z\rangle|}{\|z\|} = \frac{\|z\|^2}{\|z\|} = \|z\|$$

From the schwarz inequality,

$$|f(x)| = |\langle x,z\rangle| \leq \|z\|\|x\|$$

$$\Rightarrow \|f\| \leq \|z\|$$

$$\therefore \|f\| = \|z\|$$

11.4 SESQUILINEAR FORM (OR) FUNCTION

Definition: 11.4.1

Let H_1 and H_2 be Hilbert spaces. A function $h: H_1 \times H_2 \to K$ is called a sesquilinear form if

$(i)\ h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y) \quad \forall x_1, x_2 \in H_1, y \in H_2$

$(ii)\ h(\alpha x, y) = \alpha\ h(x, y) \quad \forall x \in H_1, y \in H_2, \alpha \in K$

$(iii)\ h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2) \quad \forall x \in H_1, y_1, y_2 \in H_2$

$(iv)\ h(x, \alpha y) = \bar{\alpha} h(x, y) \quad \forall x \in H_1, y \in H_2, \alpha \in K$

Clearly every inner product is a sesquilinear.

Definition: 11.4.2

A sesquilinear form is said to be bounded if

$\exists\ M > 0 \ni: |h(x, y)| \leq M\|x\|\|y\|$

$\forall x \in H_1, y \in H_2$, the norm of h is defined by

$$\|h\| = \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} = \sup_{\substack{x \in H_1, y \in H_2 \\ \|x\| = 1, \|y\| = 1}} |h(x,y)|$$

11.5 RIESZ'S REPRESENTATION THEOREM

Theorem: 11.5.1

Let H_1, H_2 be Hilbert spaces. If $h: H_1 \times H_2 \to K$ is a bounded sesquilinear form, then there exists unique bounded linear operator $s: H_1 \to H_2$ such that

$h(x,y) = \langle sx, y \rangle \ \forall (x,y) \in H_1 \times H_2$ with $\|h\| = \|s\|$.

Proof:

For each $x \in H$, define $g_x: H_2 \to K$ by $g_x(y) = \overline{h(x,y)} \ \ \forall y \in H_2$

Claim: g_x is a bounded linear functional on H_2

Let $y_1, y_2 \in H_2$ and $\alpha, \beta \in K$ be arbitrary

$$\begin{aligned} g_x(\alpha y_1 + \beta y_2) &= \overline{h(x, \alpha y_1 + \beta y_2)} \\ &= \overline{\bar{\alpha} h(x, y_1) + \bar{\beta} h(x, y_2)} \\ &= \alpha \overline{h(x, y_1)} + \beta \overline{h(x, y_2)} \\ &= \alpha g_x(y_1) + \beta g_x(y_2) \end{aligned}$$

$\therefore g_x$ is linear

Now for every $y \in H_2$, $|g_x(y)| = \left|\overline{h(x,y)}\right| \leq \|h\| \|x\| \|y\|$

$\Rightarrow \|g_x\| \leq \|h\| \|x\| < \infty$

$\Rightarrow g_x$ is bounded.

Then by Riesz's theorem, there exists unique z in H_2 such that $g_x(y) = \langle y, z \rangle \ \ \forall y \in H_2$.

Then, we can define $s: H_1 \to H_2$ by $sx = z$, where $z \in H$ is obtained as before.

Claim: s is a linear map

Let $x_1, x_2 \in H_1$ and $\alpha, \beta \in K$

If $sx_1 = z_1$ and $sx_2 = z_2$ then we show that $s(\alpha x_1 + \beta x_2) = \alpha z_1 + \beta z_2$.

For $y \in H_2$,

$$g_{\alpha x_1+\beta x_2}(y) = \overline{h(\alpha x_1 + \beta x_2, y)}$$

$$= \bar{\alpha}\overline{h(x_1, y)} + \bar{\beta}\overline{h(x_2, y)}$$

$$= \bar{\alpha} g_{x_1}(y) + \bar{\beta} g_{x_2}(y)$$

$$= \bar{\alpha}\langle y, z_1\rangle + \bar{\beta}\langle y, z_2\rangle$$

$$= \langle y, \alpha z_1\rangle + \langle y, \beta z_2\rangle$$

$$= \langle y, \alpha z_1 + \beta z_2\rangle$$

$$s(\alpha x_1 + \beta x_2) = \alpha z_1 + \beta z_2.$$

Claim: $\|s\| = \|h\|$

$$\|h\| = \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} \geq \sup_{\substack{x \in H_1, sx \in H_2 \\ x \neq 0, sx \neq 0}} \frac{|h(x,sx)|}{\|x\|\|sx\|}$$

$$= \sup_{\substack{x \in H_1 \\ x \neq 0, sx \neq 0}} \frac{|\langle sx, sx\rangle|}{\|x\|\|y\|} = \frac{\|sx\|^2}{\|x\|\|sx\|}$$

$$\Rightarrow \|sx\| \leq \|h\|\|x\|$$

$$\Rightarrow \quad \|s\| \leq \|h\| \ldots\ldots\ldots(1)$$

$$\|h\| = \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} = \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|h(sx,y)|}{\|sx\|\|y\|}$$

$$\leq \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{\|sx\|\|y\|}{\|x\|\|y\|} \leq \|s\| \ldots\ldots\ldots(2)$$

From (1) and (2), we have $\|s\| = \|h\|$ and hence s is a bounded linear operator such that

$$h(x, y) = \langle sx, y\rangle \quad \forall\, x \in H_1, y \in H_2$$

suppose there exists $s_1, s_2 \in H_1$ such that

$\langle s_1, y\rangle = h(x,y) = \langle s_2, y\rangle \quad \forall x \in H_1, y \in H_2$

$\Longrightarrow s_1 x = s_2 x \quad \forall x \in H_1$

$\Longrightarrow s_1 = s_2$

11.6 ADJOINT OPERATOR

Definition: 11.6.1

Let $T: H_1 \times H_2$ be a bounded linear operator, where H_1, H_2 are Hilbert spaces.

The adjoint operator T^* of T is defined by a bounded linear operator

$T^*: H_1 \times H_2$ satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall \quad \forall x \in H_1, y \in H_2$.

Theorem: 11.6.2

For every bounded linear operator $T: H_1 \times H_2$ adjoint operator T^* of T exists

and it is unique. Further $\|T\| = \|T^*\|$.

Proof:

Define a sesquilinear form $h: H_1 \times H_2 \to K$ by

$h(y,x) = \langle y, Tx\rangle \; \forall y \in H_2, x \in H_1$.

Claim: h is a sesquilinear form,

(i) Let $y_1, y_2 \in H_2 \quad \alpha, \beta \in K, \; x \in H_1$

$h(\alpha y_1 + \beta y_2, x) = \langle \alpha y_1 + \beta y_2, Tx\rangle$

$= \alpha\langle y_1, Tx\rangle + \beta\langle y_2, Tx\rangle$

$= \alpha h(y_1, x) + \beta h(y_2, x)$

(ii) Let $x_1, x_2 \in H_1 \quad \alpha, \beta \in K, \; y \in H_2$

$h(y, \alpha x_1 + \beta x_2) = \langle y, T(\alpha x_1 + \beta x_2)\rangle$

$$= \bar{\alpha}\langle y, Tx_1\rangle + \bar{\beta}\langle y, Tx_2\rangle$$

$$= \bar{\alpha}(y, Tx_1) + \bar{\beta}(y, Tx_2)$$

$\therefore$ h is a sesquilinear.

Claim: $\|h\| = \|T\|$

$$\|h\| = \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|h(y,x)|}{\|x\|\|y\|} \geq \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|\langle y, Tx\rangle|}{\|x\|\|y\|}$$

$$= \sup_{\substack{x \in H_1 \\ x \neq 0, sx \neq 0}} \frac{|\langle Tx, Tx\rangle|}{\|Tx\|\|Tx\|}$$

$\Rightarrow \|Tx\| \leq \|h\|\|x\|$

$\Rightarrow \|T\| \leq \|h\| \ldots\ldots\ldots (1)$

$$\|h\| = \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{|\langle y, Tx\rangle|}{\|x\|\|y\|} \geq \sup_{\substack{x \in H_1, y \in H_2 \\ x \neq 0, y \neq 0}} \frac{\|y\|\|Tx\|}{\|x\|\|y\|}$$

$$\leq \|T\|$$

$\Rightarrow \|h\| = \|T\| < \infty$

$\Rightarrow$ h is a bounded sesquilinear form.

Then by Riesz's representation theorem, there exists a unique bounded linear operator

$T^*: H_1 \times H_2$, such that

$h(y,x) = \langle T^*y, x\rangle \quad \forall y \in H_2, x \in H_1$

$i.e. \langle y, Tx\rangle = \langle T^*y, x\rangle \quad \forall y \in H_2, x \in H_1$

$\Rightarrow \quad \langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall y \in H_2, x \in H_1$

From $\|T^*\| = \|h\| - \|T\|$.

Theorem: 11.6.3

Let X and Y be inner product spaces. If $Q: X \to Y$ is a bounded linear operator

Such that

$(i)\langle Qx, y\rangle = 0 \quad \forall x \in X \text{ and } y \in Y$, then Q=0.

$(ii)\langle Qx, x\rangle = 0 \quad \forall x \in X$, then Q=0 provided X=Y and X is a complex inner

Product space.

Proof:

$(i)\langle Qx, y\rangle = 0 \quad \forall x \in X, y \in Y$

$\Longrightarrow \langle Qx, Qx\rangle = 0 \; \forall x \in X$

$\Longrightarrow \|Qx\|^2 = 0 \; \forall x \in X$

$\Longrightarrow Qx = 0 \; \forall x \in X$

$\Longrightarrow$ Q=0

(ii) Let $a \in \mathbb{C}$ and $x, y \in X$

$0 = \langle Q(ax + y), ax + y\rangle$

$= \langle aQx + Qy, ax + y\rangle$

$= |a|^2\langle Qx, x\rangle + a\langle Qx, y\rangle + \bar{a}\langle Qy, x\rangle + \langle Qy, y\rangle$

$\Longrightarrow a\langle Qx, y\rangle + \bar{a}\langle Qy, x\rangle = 0$

For $a = i$, $i\langle Qx, y\rangle - i\langle Qy, x\rangle = 0$

$\Longrightarrow \langle Qx, y\rangle = \langle Qy, x\rangle \ldots\ldots\ldots.(1)$

For $a = 1$, $\langle Qx, y\rangle + \langle Qy, x\rangle = 0$

$\Longrightarrow \langle Qx, y\rangle = -\langle Qy, x\rangle$

$\Longrightarrow \langle Qy, x\rangle = -\langle Qy, x\rangle$

$\Longrightarrow \langle Qy, x\rangle = 0$

By case (i), we get Q=0.

11.7 PROPERTIES OF ADJOINT OPERATORS

Theorem: 11.7.1

Let H_1 and H_2 be Hilbert spaces. Let $T: H_1 \to H_2$ and $S: H_1 \to H_2$ be bounded linear operators and 'a' be a scalar. Then

$(i) \langle T^*y, x\rangle = \langle y, Tx\rangle \quad \forall y \in H_2, x \in H_1$

$(ii) (S+T)^* = S^* + T^*$

$(iii) (\alpha T)^* = \alpha T^*$

$(iv) T^{**} = T$

$(v) \|T^*T\| = \|TT^*\| = \|T\|^2$

$(vi) T^*T = 0 \; iff \; T = 0$

$(vii) (ST)^* = T^*S^* \; provided \; H_1 = H_2$

Proof:

(i) By definition of T^*, we have

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall y \in H_2, x \in H_1$$

Applying complex conjugate operator on both sides we get

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall y \in H_2, x \in H_1$$

$$\Longrightarrow \langle y, Tx\rangle = \langle T^*y, x\rangle \quad \forall y \in H_2, x \in H_1$$

(ii) Let $y \in H_2 \; and \; x \in H_1$

$$\langle (S+T)^*, x\rangle = \langle y, (S+T)x\rangle$$

$$= \langle y, (Sx + Tx)\rangle$$

$$= \langle y, Sx\rangle + \langle y, Tx\rangle$$

$$= \langle S^*y, x\rangle + \langle T^*y, x\rangle$$

$$= \langle S^*y + T^*y, x\rangle$$

$$= \langle (S^* + T^*)y, x\rangle$$

$\therefore (S+T)^* = S^* + T^*$

(iii) Let $y \in H_2$ and $x \in H_1$

$\langle (\alpha S)^* y, x\rangle = \langle y, (\alpha S)x\rangle = \langle y, \alpha(Sx)\rangle$

$$= \bar{\alpha}\langle y, Sx\rangle = \bar{\alpha}\langle S^*y, x\rangle = \langle \bar{\alpha}S^*y, x\rangle$$

$\Longrightarrow (\alpha S)^* = \alpha S^*$

$(iv) \langle T^{**}x, y\rangle = \langle x, T^*y\rangle \ \forall\, y \in H_2, x \in H_1$

$\Longrightarrow \langle T^*y, x\rangle = \langle y, T^{**}x\rangle \ \forall\, y \in H_2, x \in H_1$

We also have $\langle T^*y, x\rangle = \langle y, Tx\rangle \ \forall\, y \in H_2, x \in H_1$

$\Longrightarrow \langle y, T^{**}x\rangle = \langle y, Tx\rangle \ \forall\, y \in H_2, x \in H_1$

$(v) First\ show\ that\ T^*T: H_1 \to H_1, TT^*: H_2 \to H_2$

$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle$

$$\leq \|x\|\|T^*Tx\| \leq \|T^*T\|\|x\|^2$$

$\|Tx\| \leq \sqrt{\|T^*T\|}\,\|x\|$

$\|T\| \leq \sqrt{\|T^*T\|} \ \ldots\ldots\ldots (1)$

$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$

$\Longrightarrow \|T\|^2 = \|T^*T\|$

Next, $\|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle y, TT^*y\rangle$

$$\leq \|y\|\|T^*Ty\| \leq \|y\|\|T^*T\|\|y\|$$

$\Longrightarrow \|T^*y\| \leq \sqrt{\|T^*T\|}\,\|y\|$

$\|T^*\| \leq \sqrt{\|T^*T\|}$

$\Longrightarrow \|T^*\|^2 = \|TT^*\|$

$\|T\|^2 = \|TT^*\| \quad (since\ \|T\| = \|T^*\|)$

$$\leq \|T\|\|T^*\| = \|T\|^2$$

$$\Longrightarrow \|T\|^2 = \|TT^*\| = \|T^*T\|$$

(vi) $TT^* = 0 \Leftrightarrow \|TT^*\| = 0 \Leftrightarrow \|T\| = 0 \Leftrightarrow T = 0$

(vii) $Let\ y \in H_2\ and\ x \in H_1$

$$\langle (ST)^*y, x\rangle = \langle y, (ST)x\rangle = \langle y, S(Tx)\rangle$$

$$= \langle S^*y, Tx\rangle = \langle T^*(S^*y), x\rangle = \langle T^*(S^*y), x\rangle$$

$$= \langle (T^*S^*)y, x\rangle$$

$$\Longrightarrow (ST)^* = T^*S^*$$

Problems:

(1) Prove that $0^* = 0$ and $I^* = I$

Solution:

Let H_1 and H_2 be Hilbert spaces,

For $x \in H_1$ and $y \in H_2$

$$\langle 0^*y, x\rangle = \langle y, 0x\rangle = \langle y, 0\rangle = 0.$$

$$\Longrightarrow 0^*y = 0\ \ \forall y \in H_2$$

$$\Longrightarrow 0^* = 0$$

Let H be a Hilbert space

For $x, y \in H, \langle Iy^*, x\rangle = \langle y, Ix\rangle = \langle y, x\rangle = \langle Ix, y\rangle$

$$\Longrightarrow I^* = I$$

11.8 CLASSIFICATION OF BOUNDED LINEAR OPERATOR

Definition: 11.8.1

Let $T: H \to H$ be a bounded linear operator, where H is a Hilbert space

(i) T is said to be self-adjoint if $T = T^*$

(ii) T is said to be unitary if $T^* = T^{-1}$

(iii) T is said to be normal if $TT^* = T^*T$

Remark:

(i) If T is unitary then T is bijection

(ii) If T is self adjoint then T is normal

(iii)If T is unitary then T is normal

Theorem: 11.8.2

Let H be a complex Hilbert space. $T: H \to H$ is self adjoint iff $\langle Tx, x\rangle$ is real $\forall x \in H$.

Proof:

Assume that T is self adjoint

$\Rightarrow T = T^*$

$\therefore \langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}\ \forall x \in H$

$\therefore \langle Tx, x\rangle$ is real $\forall x \in H$

Conversely, assume that $\langle Tx, x\rangle$ is real $\forall x \in H$

$\Rightarrow \langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \langle x, Tx\rangle \dots \dots \dots (1)$

Be definition of adjoint operator $\langle Tx, x\rangle = \langle x, T^*x\rangle \forall x \in H$

Using (1), $\langle x, Tx\rangle = \langle x, T^*x\rangle\ \forall x \in H$

$\Rightarrow \langle x, (T - T^*)x\rangle = 0\ \ \forall x \in H$

By a lemma, $T - T^* = 0 \Rightarrow T = T^*$

Therefore T is self adjoint.

Theorem: 11.8.3

Let $\{T_n\}$ be a sequence of self adjoint operators from H into H. If $T_n \to T$ as $n \to \infty$ in B(H,H)

Proof:

$\|T - T^*\| = \|T - T_n + T_n - T_n^* + T_n^* - T^*\|$

$$\leq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\|$$

$$= \|T - T_n\| + \|(T_n - T)^*\|$$

$$= 2\|T - T_n\| \to 0 \text{ as } n \to \infty$$

$\Longrightarrow \|T - T^*\| = 0 \Longrightarrow T = T^*$

Theorem: 11.8.4

Let T and S be self adjoint operators on H , TS is self adjoint iff TS=ST

Proof:

TS is self adjoint $\Leftrightarrow TS = (TS)^*$

$\Leftrightarrow TS = S^*T^* \Leftrightarrow TS = ST$

Theorem: 11.8.5

Let U and V be unitary operators on a Hilbert space. Then $\forall x \in H$

(a) U is isometry. $i.e. \|Ux\| = \|x\|, \ \forall x \in H$

$(b) U^{-1}$ is unitary (or) U^* is unitary.

(c) UV is unitary.

$(d) \|U\| = 1, if\ H \neq 0$.

Proof:

$(a) \|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle$

$$= \langle x, U^{-1}Ux\rangle = \langle x, x\rangle = \|x\|^2$$

Therefore $\|Ux\| = \|x\|$

$(b) U^{-1}$ is obviously bijection and $(U^{-1})^{-1} = U$

$\Longrightarrow (U^*)^{-1} = U = U^{**} = (U^{-1})^*$

$\Longrightarrow (U^{-1})^* = U = (U^{-1})^{-1} \Longrightarrow U^{-1}$ is unitary

$(c) (UV)^*(UV) = (U^*V^*)(UV)$

$= (V^*)(U^*U)(V)$

$= V^*IV = I$

Therefore UV is unitary.

$(d)\ \|U\| = \sup_{\substack{x \in H \\ x \neq 0}} \frac{\|Ux\|}{\|x\|} = 1$

Theorem: 11.8.6

Let T be a bounded linear operator on a complex Hilbert space H. T is unitary iff

T is isometry and onto.

Proof:

Assume that T is unitary

Then T is a bijection and T is an isometry.

$\Longrightarrow$ T is isometry and onto.

Conversely, assume that T is isometry and T is onto

T is isometry $\Longrightarrow \|Tx\| = \|x\|\ \ \forall x \in H$

$\Longrightarrow T$ is one-to-one

Therefore T is a bijection

For $x \in X$,

$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle$

$= \|Tx\|^2$

$= \|x\|^2$

$= \langle x, x\rangle$

$= \langle Ix, x\rangle\ \forall x \in H$

$\langle (T^*T - I)x, x\rangle = 0,\ \ \forall x \in H$

$T^*T - I = 0 \Rightarrow T^*T = I$

Now, $TT^* = TT^*I$

$= TT^*(TT^{-1})$

$= T(T^*T)T^{-1}$

$= T(I)T^{-1}$

$= TT^{-1} = I$

$\therefore$T is unitary.

11.9 EXERCISES

(1) Let H be a Hilbert space and let φ be non zero continuous linear functional on H. Let $M = \ker(\varphi)$. Show that M has codimension one.

(2) Let $g \in M^{\perp}$ be a unit vector such that any $y \in H$ can be written as $y = \lambda g + z$ where $z \in M$. Define $x = \varphi(g)g$. Show that x is such that $\varphi(y) = (y, x) \ \forall y \in H.$

(3) Let H be a Hilbert space and let $U: H \rightarrow H$ be a unitary operator. Show That U is an isometry. $i.e, \|Ux\| = \|x\| \ \ \forall x \in H$.

UNIT XII: ADJOINT OPERATOR IN NORMED SPACES

Structure

12.1 INTRODUCTION

This chapter contains the basis of more advanced theory of normed and banach spaces without which usefulness of these spaces and their application would be rather limited and also discuss the most important theorem of this chapter is Hahn Banach Theorem. It is an extension of linear functional on vector spaces, it also gives an example related to this theorem. Some useful results for ad joint operator will be discussed and eventually give the proof of category theorem and uniform boundedness theorem.

12.2 OBJECTIVES

Student will able to understand

Relation between adjoint operator and Hilbert adjoint operator. Identify which number is first and second category. How to extend linear functional on vector space.

12.3 HAHN-BANACH THEOREM FOR REAL VECTOR SPACE

Let X be a real vector space and P be a sub linear form on X (i.e., $P: X \to \mathbb{R}$ with

$P(x+y) \leq P(x) + p(y)\ \forall x, y \in X, P(\alpha x) = \alpha P(x),\ x \in X$). If Z is a subspace of X and $f: Z \to \mathbb{R}$ is a linear functional satisfying

$f(x) \le P(x) \forall\, x \in Z$ then there exists a linear functional $\tilde{f}: X \to \mathbb{R}$ such that $\tilde{f}(x) = f(x) \forall x \in Z$ and $\tilde{f}(x) \le P(x) \forall x \in X$.

Proof:

Let E =

$\{(g, \mathfrak{D}(g)) : \mathfrak{D}(g) \text{ is a subspace of } X \text{ with } Z \subseteq \mathfrak{D}(g),\ g: \mathfrak{D}(g) \to \mathbb{R} \text{ is linear}, g(x) = f(x)\ \forall x \in Z \text{ and } g(x) \le P(x)\ \forall x \in \mathfrak{D}(g)\}$

We define a partial order relation $\ll$ on E by $(g_1, \mathfrak{D}(g_1)) \ll (g_2, \mathfrak{D}(g_2))$ if $\mathfrak{D}(g_1) \subseteq \mathfrak{D}(g_2)$ and $g_1(x) = g_2(x)\ \forall x \in \mathfrak{D}(g)$.

<u>**Claim:**</u> << is a partial order relation on E.

First we note that $E \neq \emptyset \quad ((f, z) \in E)$

Clearly,

$(g, \mathfrak{D}(g)) \ll (g, \mathfrak{D}(g))\ \forall\, (g, \mathfrak{D}(g)) \in E$

If $(g_1, \mathfrak{D}(g_1)) \ll (g_2, \mathfrak{D}(g_2))$ and $(g_2, \mathfrak{D}(g_2)) \ll (g_1, \mathfrak{D}(g_1))$ then $\mathfrak{D}(g_1) \subseteq \mathfrak{D}(g_2), \mathfrak{D}(g_2) \subseteq \mathfrak{D}(g_1)$

$\Rightarrow \mathfrak{D}(g_1) = \mathfrak{D}(g_2)$

$\Rightarrow g_1(x) = g_2(x)\ \forall x \in \mathfrak{D}(g_1) = \mathfrak{D}(g_2)$

$\Rightarrow g_1 = g_2$

$\therefore (g_1, \mathfrak{D}(g_1)) = (g_2, \mathfrak{D}(g_2))$.

Let $(g_1, \mathfrak{D}(g_1)) \ll (g_2, \mathfrak{D}(g_2))$ and $(g_2, \mathfrak{D}(g_2)) \ll (g_3, \mathfrak{D}(g_3))$

$\Rightarrow \mathfrak{D}(g_1) \subseteq \mathfrak{D}(g_2)$ and $\mathfrak{D}(g_2) \subseteq \mathfrak{D}(g_3)$

$\Rightarrow \mathfrak{D}(g_1) \subseteq \mathfrak{D}(g_3)$

If $x \in \mathfrak{D}(g_1)$ then $g_1(x) = g_2(x)$ and $g_2(x) = g_3(x)$

$\Rightarrow g_1(x) = g_2(x) \quad \forall x \in \mathfrak{D}(g_1)$

$\therefore \ll$ is a partially ordered relation on E.

Let C be a chain in E. Then define $\mathfrak{D}(g_0) = \bigcup_{(g,\mathfrak{D}(g))\in C} \mathfrak{D}(g)$ and $g_0(x) = g(x)$ if $x \in \mathfrak{D}(g)\ \forall g \in C$.

Claim: $(g_0, \mathfrak{D}(g_0)) \in E$

Let $x, y \in \mathfrak{D}(g_0)$ and $\alpha, \beta \in \mathbb{R}$

$x \in \mathfrak{D}(g_1)$ and $y \in \mathfrak{D}(g_2)$ for some $(g_1, \mathfrak{D}(g_1)), (g_2, \mathfrak{D}(g_2)) \in C$.

Since C is a chain, without loss of generality assume that $(g_1, \mathfrak{D}(g_1)) \ll (g_2, \mathfrak{D}(g_2))$

$\Rightarrow \mathfrak{D}(g_1) \subseteq \mathfrak{D}(g_2)$ and $g_1(x) = g_2(x)\ \forall x \in \mathfrak{D}(g_1)$

$\Rightarrow \alpha x + \beta y \in \mathfrak{D}(g_2)$ and $g_0(\alpha x + \beta y) = g_2(\alpha x + \beta y)$

$= \alpha g_2(x) + \beta g_2(y)$

$= \alpha g_0(x) + \beta g_0(y)$

$\therefore$ g_0 is linear and $\mathfrak{D}(g_0)$ is a subspace of X containing Z.

By definition, g_0 is an extension of $g\ \forall\ (g, \mathfrak{D}(g)) \in C$ and hence it is an extension of f with $g_0 \leq P$.

$\therefore (g_0, \mathfrak{D}(g_0)) \in E$

Clearly, $(g_0, \mathfrak{D}(g_0))$ is an upper bound of C.

By Zorn's Lemma, E has a maximal element say $(f, \mathfrak{D}(f))$.s

Claim: $\mathfrak{D}(f) = X$, otherwise there exists $Y_1 \in X \backslash \mathfrak{D}(f)$.

Put $Y_1 = \{y + \alpha y_1 : y \in \mathfrak{D}(f), \alpha \in \mathbb{R}\}$

Define $g_1 : Y_1 \to \mathbb{R}$ by $g_1(y + \alpha y_1) = f(y) + \alpha c$ for any scalar c.

Clearly, $g_1$is a linear functional and $g_1$is an extension of f.

Since Y_1 is a subspace of X containing $\mathfrak{D}(f)$.

To prove that $(g_1, Y_1) \in E$, we have to show that $g_1(x) \leq P(x)\ \forall x \in Y$.

Let $y, z \in \mathfrak{D}(\tilde{f})$ be arbitrary.

$$\tilde{f}(y) - \tilde{f}(z) = \tilde{f}(y - z)$$

$$\le P(y - z) \qquad (\because (\tilde{f}, \mathfrak{D}(\tilde{f})) \in E)$$

$$= P(y + y_1 - y_1 - z)$$

$$\le P(y + y_1) + P(-y_1 - z)$$

$$\tilde{f}(y) - P(y + y_1) \le P(-y_1 - z) + \tilde{f}(z) \qquad \forall y, z \in \mathfrak{D}(\tilde{f})$$

$$\sup_{y \in \mathfrak{D}(\tilde{f})} \left(\tilde{f}(y) - P(y + y_1)\right) \le \inf_{z \in \mathfrak{D}(\tilde{f})} \left(P(-y_1 - z) + \tilde{f}(z)\right)$$

$$\inf_{y \in \mathfrak{D}(\tilde{f})} \left(-\tilde{f}(y) + P(y + y_1)\right) \ge \sup_{z \in \mathfrak{D}(\tilde{f})} \left(-\tilde{f}(z) - P(-y_1 - z)\right)$$

Choose C between $\inf_{y \in \mathfrak{D}(\tilde{f})} \left(-\tilde{f}(y) + P(y + y_1)\right)$ and

$\sup_{z \in \mathfrak{D}(\tilde{f})} \left(-\tilde{f}(z) - P(-y_1 - z)\right)$.

$$\therefore -\tilde{f}(z) - P(-y_1 - z) \le C \le -\tilde{f}(y) + P(y + y_1)$$

Claim: $g_1(y + \alpha y_1) \le P(y + \alpha y_1)\ \forall y \in \mathfrak{D}(\tilde{f})$ and $\alpha \in \mathbb{R}$

Case:(i) $\alpha < 0$

Put $z = y/\alpha$ in $\tilde{f}(z) \quad P(\ \ y_1 \quad z) \le C$

$$\Rightarrow -\tilde{f}(y/\alpha) - P(-y_1 - y/\alpha) \le C$$

Multiply on both sides by $-\alpha$.

$$\Rightarrow \alpha\tilde{f}(y/\alpha) - (-\alpha)P(-y_1 - y/\alpha) \le -\alpha C$$

$$\Rightarrow \tilde{f}(y) - P(\alpha y_1 + y) \le -\alpha C$$

$$\Rightarrow g_1(y + \alpha y_1) = \tilde{f}(y) + \alpha C \le P(y + \alpha y_1)$$

Case:(ii) $\alpha = 0$

$g_1(y+0y_1) = g_1(y) = \tilde{f}(y) \le P(y) = P(y+0y_1)$

Case:(iii) $\alpha > 0$

In $C \le -\tilde{f}(y) + P(y+y_1)$ replace y by $\frac{y}{\alpha}$.

$$C \le -\tilde{f}(y/\alpha) + P(y/\alpha + y_1)$$

Multiply by α on both sides, we get

$\alpha C \le -\tilde{f}(y) + P(y+\alpha y_1)$

$$g_1(y+\alpha y_1) = \tilde{f}(y) + \alpha C \le P(y+\alpha y_1)$$

$\therefore (g_1, y_1) \in E$ with $(g_1, y_1) \gg (\tilde{f}, \mathfrak{D}(\tilde{f}))$ and $Y_1 \supsetneq \mathfrak{D}(\tilde{f})$

This is the contradiction to the maximality of $(\tilde{f}, \mathfrak{D}(\tilde{f}))$ in E.

$\therefore \mathfrak{D}(\tilde{f}) = X.$

$\therefore \tilde{f}: X \to \mathbb{R}$ is a required extension of f.

12.4 ADJOINT OPERATOR IN NORMED SPACES

Let X and Y be normed spaces. If T $\in B(X,Y)$ then define $T^*: Y' \to X'$ as follows. (T^*g)x = g(Tx), $\forall x \in X$ and $\forall g \in Y'$. First we note that if f$\in T^*g$then f$\in X'$, Clearly f:x$\to k$.

Let $x_1, x_2 \in X$ and $\alpha, \beta \in k$

f($\alpha x_1 + \beta x_2$)=(T^*g) ($\alpha x_1 + \beta x_2$)

=g(T ($\alpha x_1 + \beta x_2$)

=g($\alpha T x_1 + \beta T x_2$) (since T is linear)

=$\alpha g(Tx_1) + \beta g(Tx_2)$ (since g is linear)

=$\alpha(T^*g(x_1)) + \beta(T^*g(x_2))$

=$\alpha f(x_1) + \beta f(x_2)$

$\therefore$ f is linear

$|f(x)|=|(T^*g(x))|=|g(Tx)|\leq\| g \|\| Tx \|\leq\| g \|\| T \|\| x \|$

$\Rightarrow f$ is bounded and $\| f \|\leq\| g \|\| T \|$

$\therefore$ f $\in X'$.

Theorem:12.2.1

Let X and Y be normed space if T, S $\in B(X,Y)$ then

$i)(T+S)^* = T^* + S^*$

$ii)(\alpha T)^* = \alpha T^*$

$iii)(TS)^* = S^*T^*$ when $X = Y$

$iv)I^* = I$ when $X = Y$

$v)(T^{-1})^* = (T^*)^{-1}$ if T^{-1} exists in B(X,Y)

Proof:

i) Let g$\in Y'$ and x$\in X$ be arbitrary

$((T+S)^*g)x$ =g(Tx+Sx)

= g(Tx)+g(Sx)

$=T^*g(x) + S^*g(x)$

$=(T^* + S^*)g(x)$

$\Rightarrow (T+S)^* = T^* + S^*$

ii) $(\alpha T)^*g(x) = g(\alpha T(x))$

$= g(\alpha(Tx)) = \alpha(g(Tx))$

$= \alpha(T^*g)(x) = (\alpha T^*)g(x))$

$\Rightarrow (\alpha T)^* = \alpha T^*$

iii) $((TS)^*g)(x)=g((TS)(x))$

$= g(T(Sx)) = (T^*g)(Sx)$

$= (S^*(T^*g))(x) = ((S^*T^*)g)(x)$

$\Rightarrow (TS)^* = S^*T^*$

iv) $(I^*g)(x) = g(Ix)$

$= g(x) = (Ig)(x)$

$\Rightarrow I^* = I$

v) I=$I^* = (TT^{-1})^*$

$= (T^{-1})^*T^*$

$\Rightarrow$I=I^* =$(T^{-1}T)^* = T^*(T^{-1})^*$

$= (T^*)^{-1} = (T^{-1})^*$

12.5 RELATION BETWEEN ADJOINT OPERATOR *T** AND HILBERT ADJOINT OPERATOR *T**

Let H_1, H_2 be Hilbert spaces and $T: H_1 \to H_2$ be bounded linear.

If $A_1: H_1' \to H_1$ is the operator defined by $A_1(f) = z$, where $f(x) = < x, z > \forall x \in H_1$ and $\| z \| = \| f \|$ and $A_2: H_2' \to H_2$ such that $A_2(g) = w$, where $g(y) = < y, w > \forall y \in H_2$ and $\| w \| = \| g \|$, then we claim that

$A_1 T^* A_2^{-1} = T^*$

Now, for $x \in H_1$,

$< Tx, w \geq g(Tx) = (T^*g)(x)$

$$= < x, z \geq < x, A_1 T^* A_2^{-1} w >$$

where z is such that $(T^*g)(x) = < x, z > \forall x \in H_1$ and

w is such that $g(y) = < y, w > \forall y \in H_2$

Since $z = A_1(T^*g) = A_1(T^*A_2^{-1}A_2 g) = (A_1 T^* A_2^{-1})(w)$

$\therefore A_1 T^* A_2^{-1} = T^*$.

12.6 MATRIX

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a bounded linear operator and $E = \{e_1, e_2, \dots, e_n\}$ be a basis of $\mathbb{R}^n$. Let the matrix representing T with respect to E is $T_E = (\tau_{jk})$.

Denote x by $(s_1, s_2, \dots, s_n)^t$, $y = T_E^x = (\eta_1, \eta_2, \dots, \eta_n)$

$$i.e.)\ \eta_j = \sum_{k=1}^{n} \tau_{j,k}\, s_k \quad , \quad j = 1,2,\dots,n$$

Let $F = (f_1, f_2, \dots, f_n)$ be the dual basis of E.

$i.e.)\ f_i \in \mathbb{R}^n$ and $f_i(e_j) = \delta_{i,j}\ \forall\ i,j \in \{1,2,\dots,n\}$

Now, $(T^* f_j)(x) = f_j(Tx) = f_j(y) = f_j(\sum_{k=1}^{n} \eta_k\, e_k)$

$$= \sum_{k=1}^{n} \eta_k f_j(e_k) = \eta_i$$

$$= \sum_{k=1}^{n} \tau_{j,k}\, s_k$$

$$= \sum_{k=1}^{n} \tau_{j,k}\, f_k(x)$$

Since $\quad f_k(x) = f_k = (\sum_{i=1}^{n} s_i\, e_i) = \sum_{i=1}^{n} s_i\, f_k(e_i) = s_k$

$$\Rightarrow T^* f_j = \sum_{k=1}^{n} \tau_{j,k}\, f_k \quad \forall j = 1,2,\dots,n$$

$\therefore$ The j^{th} column of $T_F^* \begin{pmatrix} \tau_{j,1} \\ \tau_{j,2} \\ \vdots \\ \tau_{j,m} \end{pmatrix}$ which is the j^{th} row vector of T_E

$$\Rightarrow T_F^* = T_E^t$$

Thoerem:12.2.2

Let X be a normed space and Y be a proper closed subspace of X. If $x_0 \in X \setminus Y$ and $\delta = \inf_{y \in Y} \| y - x_0 \|$ then there exists $f \in X'$ such that $f(y) = 0 \; \forall y \in Y, f(x_0) = \delta$ and $\| f \| = 1.$

Proof:

Let Z be the subspace of X generated by Y and x_0.

$(i.e.)\quad Z = \{y + \alpha x_0 : \alpha \in K, y \in Y\}$

Define $f_1 : Z \to K$ by $f_1(y + \alpha x_0) = \alpha\delta \;\; \forall \; y + \alpha x_0 \in Z$

Clearly f is linear and $f_1(y) = 0 \;\; \forall y \in Y, f_1(x_0) = \delta$

Claim: $\| f_1 \| = 1$

Let $z \in Z$. Then $z = y + \alpha x_0$ for some $y \in Y$ and $\alpha \in K$

$$|f_1(z)| = |\alpha|\delta = \begin{cases} 0 & if \;\; \alpha = 0 \\ |\alpha| \inf_{v \in Y} \| v - x_0 \| & if \; \alpha \neq 0 \end{cases}$$

$\leq |\alpha| \; \| \frac{-y}{\alpha} - x_0 \|$

$= \| -y - \alpha x_0 \|$

$= \| y + \alpha x_0 \| = \| z \|$

$\Rightarrow \| f_1 \| < 1$

Since $\delta = \inf_{y \in Y} \| y - x_0 \|$, there exists a sequence (y_n) in Y such that $\| y_n - x_0 \| \to \delta$ on $n \to \infty$.

Let $z_n = y_n - x_0$, then $z_n \in Z, \forall n \in \mathbb{N}$, and $f_1(z_n) = -\delta \;\; \forall n \in \mathbb{N}$.

Now, $\| f_1 \| = \sup_{0 \neq z \in Z} \frac{|f(z)|}{\|z\|} \geq \frac{|f(z_n)|}{\|z_n\|} = \frac{\delta}{\|z\|} \to 1 \;\; as \; n \to \infty$.

$\Rightarrow \| f_1 \| \geq 1$

$\Rightarrow \| f_1 \| = 1$

∴ By Hahn-Banach theorem,

There exists $f \in x'$ such that $f(x) = f_1(x)\ \forall x \in Z$ and $\| f \| = \| f_1 \|$.

For this f, we have $f(y) = 0\ \forall y \in Y$ and $f(x_0) = \delta$ and $\| f \| = 1$.

Theorem:12.2.3

Let X be a normed space. If x' is separable then x is separable.

Proof:

Let W={$f \in X' : \| f \| = 1$}

Since X' is separable , W is also separable Then W has a conutable dense subset say $\{f_n : n \in \mathbb{N}\}$. Since $\| f_n \| = 1, \| f_n \| > 1/2$ Then there exists $x_n \in X \ni \| x_n \| = 1$ and $| f_n(x_n) | > \frac{1}{2} \forall n \in \mathbb{N}$. Let Y be the closure of all finite linear combination of $\{x_n\}$ (or) the closure of span of $\{x_n\}$.

Then Y is separable.

Claim: Y=X

Suppose Y$\neq X$ Then choose $x_0 \in Y \setminus X$ and by previous theorem there exists $f \in X' \ni \| f \| = 1$ $f(y)$=0$\forall y \in X$ and $f(x_0) > 0$

For this f, We have f(x_n)=0 $\forall n \in \mathbb{N}$

½ <|f(x_n)|= $|f_n(x_n)| = |f_n(x_n) - f(x_n)|$

$$=|f_n - f)x_n| \leq \| f_n - f \| \| x_n \| = \| f_n - f \|\ \forall n \in \mathbb{N}$$

This is contradicts the fact that $\{f_n : n \in \mathbb{N}\}$ is dense in X'

∴ Y=X

∴ X is separable.

Definition:

Let (X, d) be a metric space and M$\subseteq X$

i) M is said to be **nowhere dense in X** if int $\overline{M} = \emptyset$.

ii) M is said to be of **first category (or Meager)** if M is countable union of nowhere dense sets.

iii) M is said to be of **second category (or non-meager)** if cannot be written as a countable union of nowhere dense set.

12.7 BAIRE'S CATEGORY THEOREM

If (X,d) is a complete metric space then X is second category.

Proof:

Suppose X is not a second category. Then X=$\bigcup_{k=1}^{\infty} M_k$ Where each M_k is nowhere dense set. Since M_1 is nowhere dense in X_1 $\overline{M_1} \neq x$

$\therefore \overline{M_1}^c$ is a nonempty open set in X Then there exists $p_1 \in \overline{M_1}^c$ and a radius 0<$\varepsilon_1 < \frac{1}{2} . B_1 = B(p_1, \varepsilon_1) \subseteq \overline{M_1}^c$

Since $\overline{M_2} \neq X$ and $\overline{M_2}$ does not contain any nonempty open set , $\overline{M_2}$ does not contain $B\left(p_1, \frac{\varepsilon_1}{2}\right)$ is a non empty open subset of X. Then there exists an open ball $B_2 = B(p_2, \varepsilon_2) \ni: B_2 \subseteq \overline{M_2}^c \cap B\left(p_1, \frac{\varepsilon_1}{2}\right)$ $and\ 0 < \varepsilon_2 < 1/2^2$

Proceeding like this, at the k^{th} stage we find an open ball$B^k B(p_k, \varepsilon_k) \subseteq \overline{M_k}^c \cap B\left(p_k, \frac{\varepsilon_k}{2}\right)$ and 0<$\varepsilon_k < 2^{-k}$

Claim: $\{p_k\}$ is a Cauchy sequence in X.

For m<n,
d(p_n, p_m)

$d(p_m, p_{m+1}) + d(p_m, p_{m+1}) + \cdots + d(p_{n-1}, p_n) < \frac{1}{2^m} + \cdots \frac{1}{2^{n-2}} 1\backslash$
$2^m \ \sum_{k=0}^{n=m+1} \frac{1}{2^k} \leq \frac{1}{2^m} \ \sum_{k=0}^{\infty} \frac{1}{2^k} \forall n \in \mathbb{N} = \frac{1}{2^{m+1}} \to 0\ as\ m \to n$

$\therefore \{p_k\}$ is a Cauchy sequence in X.

Since X is complete there exists p$\in X \ni: p_k \to p\ as\ k \to \infty$

Now for m<n we have

$B_n \subseteq B(p_m, \frac{1}{2}\varepsilon_m)$ and hence d(p_m, p_n)<1/2 ε_m

$\Rightarrow d(p_m, p_n) \leq d(p_m, p_n) + d(p_n, p)$ <1/2 $\varepsilon_m + d(p_n, p)$

Allowing $n \to \infty$ we have

d(p_m, p)$\leq \frac{1}{2} s_m < s_m \Rightarrow p \in B(p_m, s_m) = B_m \forall m \in \mathbb{N}$

Since $B_m \subseteq \overline{M_m}^{c} \forall m \in \mathbb{N}$ *then*

$\Rightarrow$p$\in \overline{M_m}^{c} \forall m \in \mathbb{N}$

$\Rightarrow$p$\in (\bigcap_{m=1}^{\infty} M_m)^c$

$\Rightarrow$p$\in (\bigcup_{m=1}^{\infty} M_m^c)$

$\Rightarrow$p $\notin \bigcup_{m=1}^{\infty} M_m$=X

Which is a contradiction

$\therefore$ X is a second category

12.8 UNIFORM BOUNDEDNESS THEOREM

Let X be a banach space and Y be a normed space. If $\{T_n\}$ be a sequence of bounded linear operators from X onto Y$\ni$: for each x$\in X$ there exists $c_x > 0 \ni: \| T_n x \| \leq c_x \forall n \in \mathbb{N}$ then there exists c>0$\ni: \| T_n \| \leq C \forall n \in \mathbb{N}$.

Proof:

For each A_k is closed set

Since $A_k = \bigcap_{n=1}^{\infty} \varphi_n^{-1}([-k, k])$ where $\varphi_n(x) = \| T_n x \|$ and $\forall x \in X$ *and* $\forall n \in \mathbb{N}$ and each φ_n is continuous we have A_k is a closed set.

Claim: X=$\bigcup_{k=1}^{\infty} A_k$

Let x$\in X$ then there exists $c_x > 0 \ni: \| T_n \| \leq c_x \forall n \in \mathbb{N}$ then there exists k$\in \mathbb{N} \ni: k \geq c_x$

$\Rightarrow \| T_n x \| \leq k \forall n \in \mathbb{N}$

$\Rightarrow x \in A_k$

$\Rightarrow x \in \bigcup_{k=1}^{\infty} A_k$

$\therefore X = \bigcup_{k=1}^{\infty} A_k$

Since X is complete by using Baire's category theorem

$\Rightarrow X$ is second category

Then there exists $k_n \in \mathbb{N} \ni: int\ \overline{A_{k_0}}$ Then there exists r>0 such that $B(x_0, r) \subseteq A_{k_0}$

Let $x \in X$ with $x \neq 0$ be arbitrary

Put $z = x_0 + \frac{rx}{2\|x\|}$

Then $\| z - x_0 \| = \frac{rx}{2\|x\|} < r$

$\Rightarrow z \in B(x_0, r)$

$\therefore \| T_n x \| = \| T_n \left(2 \frac{\|x\|}{r(z - x_0)}\right) \| \ \forall n \in \mathbb{N}$

$= 2\frac{\|x\|}{r}(\| T_n(z - x_0) \| \ \forall n \in \mathbb{N}$

$\leq 2\frac{\|x\|}{r}(\| T_n z \| - \| T_n x_0 \| \ \forall n \in \mathbb{N}$

$\leq 2\frac{\|x\|}{r}(k_0 + k_0) \forall n \in \mathbb{N}$

$= C\| x \| \ \forall n \in \mathbb{N}$

Where $C = \frac{\|x\|}{r} k_0$

$\therefore \| T_n \| \leq C \ \forall n \in \mathbb{N}$

12.9 EXERCISE

1. Find a meager dense subset of $\mathbb{R}$.

2. Find all rare sets in a discrete metric space in X

3. Let X be a normed space and X' its dual space. If $X \neq \{0\}$ show that X' cannot be {0}.

UNIT XIII: STRONG AND WEAK CONVERGENCE

Structure

13.1 INTRODUCTION

This chapter will be discuss the different types of convergence. This yields greater flexibility in the theory and the application of sequences and series and also give a some important results of this different types of convergence. It is useful to develop the application of investigation of spaces is often related to that of their dual spaces.

13.2 OBJECTIVE

Student will able to understand

Relation between strong and weak convergence.

How to use orthonormal sequence in Hilbert space.

Easily differentiate uniformly and strong operator convergence.

13.3 STRONG AND WEAK CONVERGENCE

Strong Convergence

Let X be a nor med space. A sequence (x_n) in X is said to be strong convergent if $\|x_n - x\| \to 0$ as $n\to\infty$ for some $x \in X$. This x is called the strong limit of (x_n).

Weakly Convergence

Let X be a nor med space. A sequence (x_n) in X is said to be weakly convergent if $f(x_n) \to f(x)$ as $n\to\infty$ for each $f \in x'$ for some $x \in X$. In this case x is called weak limit of (x_n) and its denoted by $x_n \xrightarrow{w} x$.

Lemma

Let (x_n) be a weakly convergent sequence in a nor med space. Then

i) The weak limit of (x_n) is unique.

ii) Every subsequence of (x_n) is weakly converges to same limit.

iii) $(\|x_n\|)$ is bounded.

Proof

i) Suppose $x_n \xrightarrow{w} x$ as n→∞ and $x_n \xrightarrow{w} y$ as n→∞ since the limit of convergent sequence in k is unique. We get f(x)=g(x) ∀ f∈ x' then x=y.

ii) Let $x_n \xrightarrow{w} x$ as n→∞ and (x_{n_k}) be a subsequence of (x_n). We have f(x_n)→f(x) as n→∞ ∀ f∈ x'. Since f(x_{n_k}) be a subsequence of f(x_n). f $(x_{n_k}) \rightarrow$ f(x) as k$\rightarrow \infty$.

iii) we know that of x∈ X and g_x(f) =f(x) ∀ f∈ x' then $g_x \in x'$ and $\|g_x\|$ =‖x‖. Since $x_n \xrightarrow{w} x$ as n→∞ and f(x_n)→f(x) as n→∞ $\forall n \in N$. There exists $c_f > 0$ $\ni: |f(x_n)| \leq c_f$ $\forall n \in N$. This implies that $|g_{x_n}| \leq c_f$ $\forall n \in N$. Then by uniform boundedness theorem there exists c>0 $\ni: \|g_{x_n}\| \leq c$ $\forall n \in N$. Then $\|g_{x_n}\| \leq c$ $\forall n \in N$. Therefore $\|x_n\|$ is bounded.

Results

Let X be a normed space and (x_n) be a sequence in x.

a)If $x_n \rightarrow x$ *as* $n \rightarrow \infty$ then $x_n \xrightarrow{w} x$ as n→∞.

b)Converse of a) is not true.

c)If dim X< ∞, then converse of a) is true.

Proof:

a) If $x_n \to x$ as $n \to \infty$, then $||x_n \to x|| \to 0$ as $n \to \infty$ for each $f \in x'$.

$| f(x_n) - f(x)| = |f(x_n - x)| \leq ||f|| \, ||x_n - x|| \to 0$ as $n \to \infty$

$\Rightarrow f(x_n) \to f(x)$ as $n \to \infty$ $\forall f \in x'$

$\Rightarrow x_n \xrightarrow{w} x$ as $n \to \infty$.

b) Let H be a Hilbert space with an orthonormal sequence $\{e_n\}$. We claim that $e_n \xrightarrow{w} 0$ as $n \to \infty$.

Let $f \in H'$ By Riez's theorem there exists $z \in H$ such that $f(x) = \langle x, z \rangle$ $\forall x \in H$.

By using Bessel's inequality.

$\sum_{n=1}^{\infty} | \langle e_n, z \rangle |^2 < \infty$ as $n \to \infty$

$\Rightarrow \langle e_n, z \rangle \to 0$ as $n \to \infty$

$\Rightarrow f(e_n) \to 0$ as $n \to \infty$

$\Rightarrow e_n \xrightarrow{w} 0$ as $n \to \infty$.

But $\{e_n\}$ does not strongly convergent in X.

$||e_n - e_m||^2 = \langle e_n - e_m, e_n - e_m \rangle$

$= \langle e_n, e_n \rangle - \langle e_n, e_m \rangle - \langle e_m, e_n \rangle + \langle e_m, e_m \rangle$

$= 2$

$||e_n - e_m|| = \sqrt{2}$ does not convergent to 0 as m, $n \to \infty$

$\therefore \{e_n\}$ is not a Cauchy sequence and hence it is not a convergent sequence.

c) Assume that dim X =K and $\{z, z_1, z_2 z_3 \ldots z_k\}$ be a basis of X. Let $x_n \xrightarrow{w} x$ as $n \to \infty$.

Since x_n , $x \in X$, we write

$x_n = \alpha_1(n) z_1 + \alpha_2(n) z_2 + \cdots + \alpha_k(n) z_k$ $x = \alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_k z_k$.

If $f_j: X \to K$ are defined by $f_j(z_i)=\begin{cases}1 \text{ if } i = j \\ 0 \text{ if } i \neq j\end{cases}$ Then we know that { $f_1, f_2 \ldots f_k$} are corresponding dual basis of x'. $f_j(x_n) \to f_j(x)$ as n→∞ $\forall j = 1,2,3, \ldots k$

$\Rightarrow \alpha_{j(n)} \to \alpha_j$ as n→∞ $\forall j = 1,2,3, \ldots k$

Now $||x_n - x|| = || \sum_{i=1}^{k} \alpha_{j(n)} z_i - \sum_{i=1}^{k} \alpha_i z_i||$

$= ||\sum_{i=1}^{k} (\alpha_i{}^{n} - \alpha_i) z_i||$

$\leq \sum_{i=1}^{k} |(\alpha_i{}^{n} - \alpha_i)| \, ||z_i|| \to 0$ as n→∞

$\Rightarrow x_n \to x \; as \; n \to \infty$.

Theorem 13.1.1:

Let X be a Hilbert space and (x_n) be a sequence in X $x_n \xrightarrow{w} x$ as n→∞ iff $< x_n , z > \to < x , z >$ as n→∞ $\forall z \in$ H.

Proof

Assume that $x_n \xrightarrow{w} x$ as n→∞. For each f $\in x', f(x_n) \to f(x) as \; n \to \infty$.

We know that , if z$\in H$ and f(x)= <x,z> $\forall x \in$ H. Then f $\in H'$.

$f(x_n) \to f(x) as \; n \to \infty$

$\Rightarrow < x_n , z > \to < x , z >$ as n→∞ $\forall z \in$ H

Conversely,

Assume that $< x_n , z > \to < x , z >$ as n→∞ $\forall z \in$ H. we want to prove that $x_n \xrightarrow{w} x$ as n→∞

Let f $\in H'$. By using Riez's theorem if z$\in H$ such that f(x) = < x, z > $\forall x \in$ H

$\Rightarrow < x_n , z > \to < x , z >$ as n→∞

$\Rightarrow f(x_n) \to f(x) as\, n \to \infty$

$\Rightarrow x_n \xrightarrow{w} x$ as n→∞.

Theorem 13.1.2:

Let X be a normed space and (x_n) be a sequence in X. $x_n \xrightarrow{w} x$ as n→∞ iff a) (x_n) is bounded and b) for every total subset M of x' f$(x_n) \to f(x) as\, n \to \infty \, \forall f \in M.$

Proof

If $x_n \xrightarrow{w} x$ as n→∞ by previous lemma we have proved that a) (x_n) is bounded space and b) holds from the definition of weak convergent.

Conversely, We assume that (a) and (b) holds. Let f $\in x'$ be given

Since span of M is dance in X' then there exists a sequence (f_j) in X' such that $f_j \to f \; as\, j \to \infty \; in\, X'$.

For a given $\varepsilon > 0$ there exists j_0 such that || $f_{j_0} - f$||< $\varepsilon/_{3k}$

Where k > sup{||x_n||, ||x||}

Since f$(x_n) \to f(x) as\, n \to \infty \, \forall f \in M$ and $f_{j_0} \in span\, M$,

$f_{j_0}(x_0) \to f_{j_0}(x)$ as n→∞.

For some $\varepsilon > 0 \; there\; exists\; n \in N \; \ni |f_{j_0}(x_0) - f_{j_0}(x)| <$ $\varepsilon/_3$

For n≥ N,

|f(x_n)-f(x)| $\leq \; |f(x_n) - f_{j_0}(x_0) + f_{j_0}(x_0) - f_{j_0}(x) + f_{j_0}(x) - f(x)|$

$\leq \; |f(x_n) - f_{j_0}(x_0)| + |f_{j_0}(x_0) - f_{j_0}(x)| + |f_{j_0}(x) f(x)|$

$\leq ||f - f_{j_0}|| \; ||x_n|| + |f_{j_0}(x_n) - f_{j_0}(x_0)| + ||f_{j_0} - f|| \; ||x||$

$< {\varepsilon k}/_{3k} + {\varepsilon}/_{3} + {\varepsilon k}/_{3k} = \varepsilon$

Hence f$(x_n) \to f(x) as\, n \to \infty \Rightarrow x_n \xrightarrow{w} x$ as n→∞.

13.4 CONVERGENCE OF SEQUENCES OF OPERATOR AND FUNCTIONALS

Definition

Let T_n , $T \in B(X,Y)\ \forall n \in N$ where X and Y are normed space.

i) (T_n) uniformly operator converges to T if || T_n-T||→0 as n→∞.

ii) (T_n) strongly operator converges to T if || $T_n x$-Tx||→0 as n→∞ for each x$\in X$.

iii) (T_n) weakly operator converges to T if ||f($T_n x$)-f(Tx)||→0 as n→∞ for each x$\in X$ for every f$\in Y'$.

Result

Let X and Y be normed space . T_n , T ∈B(X, Y) $\forall n \in N$.

(T_n) uniformly operator converges to T ⇒(T_n) strongly operator converges to T⇒)(T_n) weakly operator converges to T.

Proof

Let x ∈ X be arbitrary

If || T_n-T||→0 Then if || $T_n x$-Tx||≤ $\|T_n - T\| \to 0$ as n→∞.

For each f $\in Y'$ ||f($T_n x$)-f(Tx)|| ≤||f|| $T_n x$-Tx||→0 as n→∞ .

Remark

Converse of above results is not true.

Counter example

(T_n) strongly convergences to T but (T_n) does not uniformly operator convergences to T.

Solution

Take X = Y =l^2 For each n $\in N$ define

$T_n(x_1, x_2, x_3, \ldots)$=(0,0,0….0, $x_{n+1}, x_{n+2}, \ldots$) $\forall x_i \in l^2$

Prove that (T_n) is linear

$$||T_n(x_1,x_2,x_3,\ldots)||_2 = (\sum_{k=n+1}^{\infty}|x_k|^2)^{\frac{1}{2}}$$

$$\leq ||(x_1,x_2,x_3,\ldots)||_2$$

$\Rightarrow ||T_n|| \leq 1$

$||T_n||$=Sup $\frac{||T_n(x)||_2}{||x||_2}\frac{||T_n(x)||_2}{||x||_2}$

$\geq \frac{||T_n(e_{n+1})||_2}{||e_{n+1}||_2}$ Where e_{n+1} =(0,0,0…..1,0)

$= \frac{||e_{n+1}||_2}{||e_{n+1}||_2}$ =1

$\Rightarrow ||T_n|| \geq 1$

$=||T_n||$ =1 $\forall n \in N$.

Let $(x_k) \in l^2$ be arbitrary

$Now, ||T_n(x_k)||^2 = ||\ (0,0,0\ \ldots.0,x_{n+1},x_{n+2},\ldots\ldots)\ ||^2$

$=\sum_{k=n+1}^{\infty}|x_k|^2 \to 0$ as $n\to\infty$

$\Rightarrow ||T_n(x_k)|| \to 0\ as\ n \to \infty$

$\Rightarrow T_n(x_k) \to 0\ as\ n \to \infty$

But $||\ T_n||=1$ for every $n\in N \Rightarrow\ ||\ T_n||$ does not converges to 0 as $n\to\infty$.

Therefore (T_n) strongly operator converges to zero but not uniformly converges to zero.

Example 2

Let X=Y=l^2 Define $T_n: l^2 \to l^2$ by $T_n(x_1,x_2,x_3\ldots)$ =(0,0……..0,x_1,x_2,……) $\forall n \in N$. Clearly $\forall n \in N$

Claim: (T_n) weakly operator converges to zero. Let $x_k \in l^2$ and f $\in l^2$ By riesz's theorem $z_k \in l^2$ such that f(y_k)=$\sum_{k=1}^{\infty} y_k z_k, \forall y_k \in l^2$.

$|f(T_n(x_k))| = |\ \sum_{k=1}^{\infty} x_k z_{n+k}\ |$

$$\leq \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}} \quad \left(\sum_{k=1}^{\infty} |z_{n+k}|^2\right)^{\frac{1}{2}}$$

$$= ||(x_k)||_2 \sum_{k=1}^{\infty} |z_j|^2)^{\frac{1}{2}} \to \infty \; as \; n \to \infty$$

f($T_n(x_k)$) $\to \infty$ $as\; n \to \infty \Rightarrow (T_n)$ weakly operator converges to zero. But (T_n) denote strongly operator converges to zero.

consider $e_1 = (1,0,0, \ldots) \in l^2$

For norm $||T_n e_1 - T_m e_1|| = ||e_{n+1} - e_{m+1}||$

$= \sqrt{2}$ does not converges to 0 as n$\to\infty$

Hence our claim holds.

Lemma

Let X be a banch space and Y be a normed space. If $T_n \in B(X, Y)$ and (T_n) strongly operator converges to T then T $\in B(X, Y)$.

Proof:

For every $\in X$, $T_n x \to Tx \; as \; n \to \infty \; in \; Y$

$\Rightarrow$ ($T_n x$) is bounded in Y for each x$\in X$

By using uniform boundedness theorem

c>0 such that $||T_n|| \leq C \; \forall \; n \in N.$

$\Rightarrow$||Tx||= || $\lim_{n\to\infty} T_n x$ || = $\lim_{n\to\infty} ||T_n x||$

$\leq \lim_{n\to\infty} ||T_n||\,||x|| \leq$ c ||x|| $\forall \; x \in X$

Therefore T $\in B(X, Y)$.

Definition:

Let $f_n \in X' \forall n \in \mathbb{N}.$ we say that (f_n) $Weak^*$ converges to f in X' if for each x$\in X$, $f_n(x)$ $\to$f(x) as n$\to\infty$.

Theorem13.2.2:

Let X and Y be banach space $T_n \in B(X,Y) \quad \forall n \in N$. (T_n) be strongly operator converge to T iff

A) $\|T_n\|$ is bounded

B) $(T_n x)$ is Cauchy in Y for each x belongs to total subsets of X

Proof

If (T_n) strongly operator converges to T

⇒ A) and B) holds (obviously)

Assume A) and B)

Let $M \subseteq X$ be a total set

⇒ span of M is dense in X.

Let $x \in X$ using A) choose c>0 such that $\| T_n \| \leq C \; \forall n \in \mathbb{N}$

Choose $y \in$ span M , such that $\| x-y \| < \frac{\epsilon}{3c}$ …………(1)

Since $y \in$ span M using B) $(T_n y)$ is Cauchy in Y.

Given $\varepsilon > 0$ there exists $N \in \mathbb{N} \ni$: $\| T_n x - T_m y\| < \frac{\varepsilon}{3}$

$\forall m, n \in \mathbb{N}$………..(2)

For $m, n \in \mathbb{N}$

$\| T_n x - T_m y\| \leq \|T_n x - T_n y\| + \| T_n y - T_m y\| + \| T_m y - T_m x\|$

$< \|T_n\| \, \| x- y\| + \frac{\varepsilon}{3} + \|T_m\| \, \|y-x\|$

$< c \, \frac{\varepsilon}{3c} + \frac{\varepsilon}{3} + c \, \frac{\varepsilon}{3c} = \varepsilon.$

∴ $(T_n x)$ is Cauchy in Y

Since Y is complete, $(T_n x)$ is converges in Y.

13.5 EXERCISE

1. Show that any closed subspace Y of a normed space X contains the limits of all weakly convergent sequence of its elements.
2. Let A be a set in a normed space X such that every nonempty subset of A contains a weak cauchy sequence. Show that A is bounded.
3. A normed space X is said to be weakly complete if each weak Cauchy sequence in X converges weakly in X. If X is reflexive, show that X is weakly complete.

UNIT XIV: OPEN MAPPING THEOREM

Structure

14.1 INTRODUCTION

This chapter concerned with open mappings. These mappings such that the image of every open set is open set and also discuss with bounded linear operator is an open mapping. Then will prove the closed graph theorem and also give examples of related the open and closed graph theorem. It is important closed graph theorem which states the sufficient conditions under which a closed linear operator on a banach space is bounded.

14.2 OBJECTIVE

Student will able to understand

- Compute the mapping between the any two spaces.
- Identify by which one is surjective, injective and continuous mapping.
- Differentiate Bounded and closed linear operator.

14.3 OPEN MAPPING THEOREM

Let X and Y be Banach spaces. If T is a bounded linear operator from X onto Y then T is an open mapping.

Proof:

First we prove the following lemma.

Lemma:

If X and Y are Banach spaces and $T \in B(X, Y)$ is onto then $T(B(0,1))$ contains a neighborhood of 0 in Y.

Proof of the lemma:

Let B_0=B(0,1) and B_n = B(0, 2^{-n}) $\forall\, n \in \mathbb{N}$

Claim:1 $\overline{T(B_1)}$ Contain on open ball B^*

Let $x \in X$. If $k > 2\|x\|$, then $\frac{x}{k} \in B_1$ and hence $x \in kB_1 \Rightarrow X = \bigcap_{k=1}^{\infty} kB_1$

Since T is linear and onto we have,

Y=T(x)=T($\bigcup KB_1$) = $\bigcup KTB_1$,

Since Y is complete by Baire's category theorem int $\overline{KT(B_1)} \neq \phi$ ⇒K int

$\overline{T(B_1)} \neq \phi \Rightarrow \text{int}\,\overline{T(B_1)} \neq \phi$

Let $y_0 \in \text{int}\,\overline{T(B_1)}$

Then $\varepsilon > 0 \ni: B^* = B(y_0, \varepsilon) \subseteq \overline{T(B_1)}$

Claim 1 is holds.

Claim:2 For each n$\in \mathbb{N}$, $\overline{T(B_n)}$ contains a neighborhood V_n of o.

Using claim 1 , we have B^*-y_0 = $B(0,\varepsilon) \subseteq \overline{T(B_1)}$ -y_0, First we show that

$\overline{T(B_1)}$ -y_0 $\subseteq \overline{T(B_0)}$. Let y$\in \overline{T(B_1)}$ -y_0. Then y+$y_0 \in \overline{T(B_1)}$

We Know that $y_0 \in \overline{T(B_1)}$ also. Then there exists (z_n) and (w_n) From $B_1 \ni:$

T$z_n \to y + y_0$ as n→ ∞, T$w_n \to y + y_0$ as n→ ∞ in $\overline{T(B_1)}$.

⇒T(z_n-y_n) =Tz_n-T$y_n \to y + y_0 - y_0$=y as n→ ∞ and

$\| z_n\text{-}y_n \| \leq \| z_n \| + \| y_n \| < 1/2 + 1/2 = 1$

⇒ z_n-$y_n \in B_0\ \forall\, n \in \mathbb{N}$

⇒y∈ $\overline{T(B_0)}$

⇒ $\overline{T(B_1)}$ - $y_0 \subseteq \overline{T(B_0)}$

⇒ B(o,ε)$\subseteq \overline{T(B_0)}$

⇒2^{-n}B(o,ε) ⊂ $2^{-n}\overline{T(B_0)}$ = $\overline{2^{-n}T(B_0)}$

$$= T(2^{-n}B_0) = T(B_n) \quad \forall n \in \mathbb{N}$$

If V_n= B(0, $\frac{\varepsilon}{2^n}$), Then $V_n) \subseteq \overline{T(B_n)}$, $\forall n \in \mathbb{N}$.

∴ Claim 2 is holds.

Claim:3 T(B_0) Contains a neighborhood of 0.

Let y∈ $\overline{T(B_1)}$. For $\frac{\varepsilon}{4}$>0 there exists $x_1 \in B_1 \ni: \| y - Tx_1 \| < \frac{\varepsilon}{4}$

⇒ y$-Tx_1 \in V_2 \subseteq \overline{T(B_2)}$

For $\frac{\varepsilon}{8}$>0 there exists $x_2 \in B_2 \ni: \| y - Tx_1 - Tx_2 \| < \frac{\varepsilon}{8}$

⇒ y$-Tx_1 - Tx_2 \in V_3 \subseteq \overline{T(B_3)}$

Proceeding like this at then^{th} stage there exists $x_n \in B_n$ such that

$\| y - Tx_1 - Tx_2 - \cdots - Tx_n \| < \frac{\varepsilon}{2^{n+1}}$

$y - Tx_1 - Tx_2 - \cdots - Tx_n \in V_n \subseteq \overline{T(B_n)}$.

∴ Claim 3 is holds.

Claim 4: Let $S_n = x_1 + x_2 + \cdots + x_n \ \forall n \in \mathbb{N}$

(S_n) is a Cauchy sequence in X.

For n>m, $\| S_n - S_m \| = \| \sum_{k=m+1}^{n} x_k \|$

$$\leq \sum_{k=m+1}^{n} \| x_k \|$$

$$< \sum_{k=m+1}^{n} \frac{1}{2^k}$$

$$< \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}} \to 0 \ as \ n \to \infty$$

∴ (s_n) is cauchy in X.

Since X is complete there exists x∈ $X \ni: s_n \to x \ as \ n \to \infty$

Now, Tx=T ($\lim_{n\to\infty} s_n$)= $\lim_{n\to\infty} T s_n = y$

Since $\| s_n \| = \| x_1 + x_2 + \cdots + x_n \| \leq \| x_1 \| + \| x_2 \| + \cdots + \| x_n \|$

$$< 1 + {}^1/_2 + {}^1/_{2^2} + \cdots + {}^1/_{2^n}$$

$< \sum_{k=1}^{\infty} {}^{1}/_{2^k} = {}^{1}/_{2} \Rightarrow x \in B_0$

$\Rightarrow$ y= Tx$\in T(B_0)$ Since y$\in \overline{T(B_1)}$ is arbitrary $\overline{T(B_1)} \subseteq T(B_0)$

Using claim 3 $V_1 \subseteq \overline{T(B_1)}$

$\therefore$ $T(B_0)$ contains a neighborhood of O.

Proof:

Let U be an open subset of X. To prove T (U) is open in Y, let $y_0 \in T(U)$

Then $y_0 = Tx_0$ for some $x_0 \in U$.

Since U is open in X there exists r$> 0 \ni: B(x_0, r) \subseteq U$.

From the lemma we have B(0,$\frac{\varepsilon}{2}$) = $V_1 \subseteq$ T(B(0,1))

Now we claim that B($y_0, \frac{r\varepsilon}{2}$) ⊆T(U)

B($y_0, \frac{r\varepsilon}{2}$)= $y_0 + r$ B(0,$\frac{\varepsilon}{2}$)

⊆ T$x_0 + rT$ (B(0,1))

= T($x_0 + rB(0,1)$)

= T(B(x_0, r))

⊆T(U)

$\therefore T(U)$ is open in X

⇒T is open map.

Remark:

If X and Y are topological spaces, if f : X→Y is one to one and onto , f is open then $f^{-1}: Y \to X$ is continuous.

Definition: (Product of two normed spaces)

Let X and Y be normed spaces, Clearly X$\times Y$ is a vector space with respect to addition and scalar multiplication defined by

(x_1, y_1)+(x_2, y_2)=$(x_1 + x_2, y_1 + y_2)$

$\lambda(x,y) = (\lambda x, \lambda y)$ (x_1, y_1),(x_2, y_2),(x,y)∈ X$\times Y$, $\lambda \in K(\mathbb{R}\ or\ \mathbb{C})$

Define $\|.\|: X \times Y \to \mathbb{R}$ by $\| (x,y) \| = \| x \| + \| y \|$ $\forall (x,y) \in X \times Y$

Claim:

$\|.\|$ is a norm on $X\times Y$

i) $\|(x,y)\|=\|x\|+\|y\|\geq 0\ \forall(x,y)\in X\times Y$

ii) $\|(x,y)\|=0$ iff $\|x\|+\|y\|=0$ iff $\|x\|=0,\|y\|=0$ iff x=0,y=0 iff (x,y)=(0,0).

iii) $\|\lambda(x,y)\|=\|(\lambda x,\lambda y)\|=\|\lambda x\|+\|\lambda y\|=|\lambda|(\|x\|+\|y\|)$
$=|\lambda|\ \|(x,y)\|$

iv) $\|(x_1,y_1)+(x_2,y_2)\|=\|(x_1+x_2,y_1+y_2)\|=\|(x_1+x_2)\|+\|(y_1+y_2)\|\leq\|x_1\|+\|x_2\|+\|y_1\|+\|y_2\|=$
$\|(x_1,y_1)\|+\|(x_2,y_2)\|$

$\therefore$ $\|.\|$ is a norm on $X\times Y$.

Definition:

Let D(T) be a subspace of a normed space X and Y be a normed space. A linear operator T:D(T)$\rightarrow Y$ is said to be a closed linear operator of its graph.

$g_T=\{(x,Tx): x\in D(T)\}$ is closed in $X\times Y$.

14.4 CLOSED GRAPH THEOREM

Let X and Y be banach spces and D(T) be a closed subspace of X. if T:D(T)$\rightarrow Y$ is closed linear then bounded.

Proof:

We know that $X\times Y$ is a normed linear space with $\|x\|+\|y\|=\|(x,y)\|\ \ \forall(x,y)\in X\times Y$. First we show that $X\times Y$ is a banach space of X and Y are banach spaces. Let $(x_n,y_n)\in X\times Y$ be a Cauchy sequence in $\times Y$. Given $\varepsilon<0$ there exists

N$\in\mathbb{N}\ni:\|(x_n,y_n)-(x_m,y_m)\|<\varepsilon\ \forall m,n\geq N$

$\Rightarrow\|x_n-y_n\|+\|y_n-y_m\|<\varepsilon\ \forall\ m,n\geq N$

$\Rightarrow\|x_n-x_m\|<\varepsilon,\|y_n-y_m\|<\varepsilon\ \forall m,n\geq N$

$\Rightarrow (x_n)$ and (y_n) are Cauchy sequence in X and Y respectively. Since X and Y are complete , there exists x$\in X$, $y \in Y \ni: x_n \to x, y_n \to y$ as $n \to \infty$

$\Rightarrow \| (x_n,y_n) - (x,y) \| = \| x_n - x \| + \| y_n - y \| \to 0$ as $n \to \infty$.

$\Rightarrow (x_n,y_n)$ converges in $X \times Y$

$\therefore X \times Y$ is a banach space.

Define the projection map P:$\mathcal{G}(T) \to X$

P((x,Tx))=x $\forall (x,Tx) \in \mathcal{G}(T)$

Claim: P : $\mathcal{G}(T) \to X$ is a bijection.

If x $\in X$ then (x,Tx)$\in \mathcal{G}(T)$ and P(x,Tx) = x

$\Rightarrow P$ is onto.

If P(x,Tx)=P(y,Ty) then x = y

$\Rightarrow (x,Tx) = (y,Ty)$

$\therefore P$ is one to one

Claim: P is bounded

$$\| (Tx,Ty) \| = \| x \| \leq \| x \| + \| Tx \| = \| (x,Tx) \|$$

$\therefore P$ is bounded.

Clearly P is a linear map.

Sine $\mathcal{G}(T)$ is a clsed subspace of $X \times Y$, $\mathcal{G}(T)$ is also a banach space.

$\therefore$ By open mapping theorem

$\Rightarrow p^{-1}$ is continuous

i.e) $\| p(x) \| \leq M \| x \| \ \forall x \in X$, for some M>0

$\| Tx \| \leq \| x \| + \| Tx \| = \| (x,Tx) \| = \| p^{-1}(x) \| \leq M \| x \| \ \forall x \in X$.

$\therefore$ T is bounded.

Theorem:14.2.1

Let X and Y be normed space and $\mathcal{D}(T)$ be a subspace of X . A linear operator T:$\mathcal{D}(T) \to Y$ is closed linear iff T has the following property.

Whenever $x_n \to x$, T$x_n \to y$ as n$\to \infty$ with $x_n \in \mathcal{D}(T)$ we have $x \in \mathcal{D}(T)$ and Tx=y.

Proof:

T is a closed linear $\Leftrightarrow \mathcal{G}(T)$ is closed in $X \times Y$ $\Leftrightarrow$whenever $z_n \in \mathcal{G}(T)$ and $z_n \to z \text{ in } X \times Y \text{ as } n \to \infty$ then $z_n \in \mathcal{G}(T)$ $\Leftrightarrow$ whenever $(x_n, Tx_n) \to (x, y) \text{ as } n \to \infty \text{ in } X \times Y$ and $x_n \in \mathcal{D}(T) \forall n,\ x \in \mathcal{D}(T) \text{ and }$ y=Tx$\Leftrightarrow$ whenever $x_n \to x, Tx_n \to y \text{ as } n \to \infty$ and $x_n \in \mathcal{D}(T)\ \forall n$ then $x \in \mathcal{D}(T)$ and Tx=y.

Examples:

1**. Closed linear operator need not be a bounded linear operator**.

Solution:

Let X = Y=C([0,1]) and $\mathcal{D}(T) =$ The set of all continuously differentiable function on [0,1]. Clearly $\mathcal{D}(T)$ is a subspace of X. we know that the differential operator $\mathcal{D}$ is not bounded.

To prove $\mathcal{D}$ is closed on $\mathcal{D}(T)$.

Let $x_n \in \mathcal{D}(T) \forall n \ni x_n \to x \text{ and } x_n' \to y \text{ as } n \to \infty.$ Since (x_n') is uniformly converges on [0,1] and $(x_n(t))$ converges for any t$\in [0,1]$ by theorem $x' = \lim_{n \to \infty} x_n' = y.$ Since (x_n') converges uniformly to y and x_n' is continuous $\forall n \in \mathbb{N}.$

$\Rightarrow y = x'$ is continuous

$\Rightarrow x \in \mathcal{D}(T)$ and y=x'

$\therefore \mathcal{D}$ is closed linear.

2. Bounded linear operator need not be a closed linear operator.

Solution:

Let X be a normed space and $\mathcal{D}(T)$ be a dense proper subspace of X. $\mathcal{D}(T)$ be a dense proper subspace of X.

i.e) $\mathcal{D}(T) \subsetneq X$ and $\overline{\mathcal{D}(T)}$ =X.

If T = $\mathcal{D}(T) \to X$ is the identity operator then T is bounded. But T is not closed. Let x$\in X \setminus \mathcal{D}(T)$. Then there exists (x_n) is $\mathcal{D}(T) \ni: x_n \to x$ as n$\to \infty$. Since T is continuous T$x_n \to Tx \text{ as } n \to \infty$. But x$\notin \mathcal{D}(T) \Rightarrow T$ is not closed.

Theorem:14.2.2

Let X, Y be normed spaces, $\mathcal{D}(T) \subseteq X$ and T be a bounded linear operator from $\mathcal{D}(T) \text{ into } Y$. Then

i) If $\mathcal{D}(T)$ is closed, Then T is closed

ii) If T is closed and Y is complete then $\mathcal{D}(T)$ is closed.

Proof:

i) Let $x_n \in \mathcal{D}(T) \ni: x_n \to x \text{ and } Tx_n \to y \text{ as } n \to \infty$. Then

$x \in \overline{\mathcal{D}(T)}$ and T is continuous implies T$x_n \to Tx$ as n$\to \infty$

$\Rightarrow Tx = y$. T is closed.

ii) Let $x \in \overline{\mathcal{D}(T)}$ Then there exists a sequence (x_n) from

$\mathcal{D}(T) \ni: x_n \to x \text{ as } n \to \infty$

Claim:

(Tx_n) is Cauchy in Y. For every m,n$\in \mathbb{N}$.

$\| Tx_n - Tx_m \| \leq \| T \| \| x_n - x_m \| \to 0 \text{ as } m,n \to \infty$

$\therefore (Tx_n)$ is Cauchy in Y

Since y is complete (Tx_n) converges. Since T is closed by a theorem x$\in \mathcal{D}(T)$ and $\lim_{n\to\infty} Tx_n = Tx \Rightarrow \overline{\mathcal{D}(T)} \subseteq \mathcal{D}(T) \Rightarrow \mathcal{D}(T)$ is closed.

14.5 EXERCISE

1. Let X and Y be banach spaces and T:X$\to Y$ an injective bounded linear operator. Show that $T^{-1}: \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.
2. Show that an open mapping need not map closed sets onto closed sets.
3. Let X and Y be normed spaces. If $T_1: X \to Y$ is a closed linear operator and $T_2 \in B(X,Y)$ show that $T_1 + T_2$ is a closed linear operator.