F–2114

Sub. Code
7PMA1C1

M.Phil. DEGREE EXAMINATION, APRIL 2019

First Semester

Mathematics

RESEARCH METHODOLOGY

(CBCS – 2017 onwards)

Time : 3 Hours Maximum : 75 Marks

Part A (5 × 5 = 25)

Answer any **five** questions.

1. Distinguish between exploratory research and conclusive research.

2. Give a sample table of contents of an algorithmic research report.

3. Define the following terms. Give an example for each :

   (a) Submodule

   (b) Annihilator of an R-module M

   (c) Exact sequence.

4. Let $M = \oplus\sum_{i=1}^{n} M_i$ and $N = \oplus\sum_{j=1}^{m} N_j$. Prove that $M \otimes N \simeq \oplus\sum_{i,\, j}(M_i \otimes N_j)$.

5. Define comaximal with an example. Let $I_1, I_2, ...I_n$ be mutually comaximal ideals of R. Prove that $\bigcap_1^n I_i = \prod_1^n I_i$.

6. With the usual notations, prove that R_S is a flat R-module.

7. Write down the coercivity condition on $I[\cdot]$.

8. Narrate the Harmonic maps.

Part B (5 × 10 = 50)

Answer **all** questions choosing either (a) or (b).

9. (a) Enumerate the following terms :
   (i) Mathematical model ;
   (ii) Algorithmic research ;
   (iii) Data collection ;
   (iv) Interpretation of results.

Or

(b) (i) What are the types of report? Explain them in brief.
   (ii) Discuss the guidelines for preparing bibliography.

10. (a) Show that an R-module $M = M_1 \oplus M_2 \oplus ..... \oplus M_n$ if and only if
   (i) $M = M_1 + M_2 + ... + M_n$ and
   (ii) $M_i \cap (M_1 + M_2 + ... + M_{i-1} + M_{i+1} + ... + M_n) = 0$ for all $i, 1 \le i \le n$.

Or

(b) (i) If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is a split exact sequence, then prove that $M \simeq M' \oplus M''$.

(ii) Define the following terms. Give an example for each : Flat and faithfully flat.

11. (a) (i) Define a local ring with an example. Prove that R is a local ring if and only if it has a unique maximal ideal.

(ii) State and prove Nakayama lemma.

Or

(b) Prove that the canonical map $f : R_s \underset{R}{\otimes} M \to M_s$ defined by $f\left[(a/s) \otimes x\right] \to (ax/s)$ is well defined and is an isomorphism of R_s-modules.

12. (a) (i) State and prove Divergence-free rows lemma.

(ii) What is meant by weakly lower semicontinuous function?

Or

(b) State and prove weak lower semicontinuity theory.

13. (a) State and prove the second derivatives for minimizers theorem.

Or

(b) State and prove the Deformation theorem.

F–2115

Sub. Code
7PMA1C2

M.Phil. DEGREE EXAMINATION, APRIL 2019

First Semester

Mathematics

FUNCTIONAL ANALYSIS

(CBCS – 2017 onwards)

Time : 3 Hours — Maximum : 75 Marks

Part A $(5 \times 5 = 25)$

Answer any **five** questions.

1. (a) Define a normed space with an example.

   (b) What is meant by topological vector space?

2. If X is a complex topological vector space and $f : \mathbb{C}^n \to X$ is linear, then prove that f is continuous.

3. Define the following terms :

   (a) Cauchy sequence

   (b) Seminorm

   (c) Quotient norm.

4. Define the quotient space. Also prove that L^p is a locally bounded F – space.

5. State and prove the Banach-Steinhaus theorem.

6. If f is a continuous linear functional on a subspace M of locally convex space X, then prove that there exists $\wedge \in X^*$ such that $\wedge = f$ on M.

7. Suppose X and Y are Banach spaces, and $T \in \mathcal{B}(X, Y)$. Prove the following :

(a) $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}$

(b) $\mathcal{N}(T) = {}^{\perp}\mathcal{R}(T^*)$

8. Let M be a closed subspace of a topological vector space X. If X is locally convex and $\dim M < \infty$, then prove that M is complemented in X.

Part B (5 × 10 = 50)

Answer **all** questions choosing either (a) or (b).

9. (a) Let $\wedge$ be a linear functional on a topological vector space X. Assume $\wedge x \neq 0$ for some $x \in X$. Prove that each of the following four properties implies the other three :

(i) $\wedge$ is continuous

(ii) the null space $\mathcal{N}(\Lambda)$ is closed

(iii) $\mathcal{N}(\Lambda)$ is not dense in X

(iv) $\wedge$ is bounded in some neighbourhood V of 0.

Or

(b) If X is a topological vector space with a countable local base, then prove that there is a metric d on X such that

(i) d is compatible with the topology of X

(ii) the open balls centered at 0 are balanced, and

(iii) d is invariant : $d(x+z, y+z) = d(x,y)$ for $x,y,z \in X$

If in addition, X is locally convex, then d can be chosen so as to satisfy (i), (ii), (iii) and also

(iv) all open balls are convex

10. (a) Suppose Y is a subspace of a topological vector space X and Y is an F - space (in the topology inherited form X). Prove that Y is a closed subspace of X.

Or

(b) (i) Prove that a topological vector space X is normable if and only if its origin has a convex bounded neighbourhood.

(ii) State the Heine-Borel property.

11. (a) State and prove the open mapping theorem.

Or

(b) State and prove the closed graph theorem.

12. (a) Suppose :

(i) M is a subspace of a real vector space X,

(ii) $p : X \to R$ satisfies

$p(x+y) \le p(x) + p(y)$ and $p(tx) = p(x)$ if $x \in X$, $y \in X$, $t \ge 0$.

(iii) $f : M \to R$ is linear and $f(x) \le p(x)$ on M prove that there exists a linear $\wedge : X \to R$ such that $\wedge x = f(x)$ $(x \in M)$ and $-p(-x) \le \wedge x \le p(x) (x \in X)$.

Or

(b) State and prove the Banach-Alaoglu theorem.

13. (a) Suppose X and Y are Banach spaces, and $T \in \mathcal{B}(X, Y)$. Prove that

(i) $\mathcal{R}(T) = Y$ if and only if

(ii) T^* is one-to-one and $\mathcal{R}(T^*)$ is norm-closed.

Or

(b) Suppose X and Y are Banach spaces and $T \in \mathcal{B}(X,Y)$. Prove that T is compact if and only if T^* is compact.

F–2116

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7PMA2C1

M.Phil. DEGREE EXAMINATION, APRIL 2019

Second Semester

Mathematics

ANALYSIS

(CBCS – 2017 onwards)

Time : 3 Hours Maximum : 75 Marks

Part A (5 × 5 = 25)

Answer any **five** questions.

1. Let u and v be real measurable functions on a measurable space X, let ϕ be a continuous mapping of the plane into a topological space Y, and define $h(x) = \phi(u(x), v(x))$ for $x \in X$. Prove that $h : X \to Y$ is measurable.

2. Suppose $\mu(X) < \infty, f \in L'(\mu)$, S is a closed set in the complex plane, and the averages $A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu$ lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Prove that $f(x) \in S$ for almost all $x \in X$.

3. Define the following terms:
   (a) Hausdorff space;
   (b) Locally compact;
   (c) Borel measure.

4. State and prove the Lusin's theorem.

5. For $1 \leq p \leq \infty$, prove that $C_e(X)$ dense in $L^P(\mu)$.

6. Derive the Schwarx inequality.

7. Show that every orthonormal set B in a Hilbert space H is contained in a maximal orthonormal set in H.

8. For every $x \in A$, prove that $\sigma(x)$ is compact and not empty.

Part B (5 × 10 = 50)

Answer **all** questions choosing either (a) or (b).

9. (a) Let μ be a positive measure on a $\sigma-$algebra $\mathfrak{M}$. Prove the following:

(i) $\mu(\phi) = 0$.

(ii) $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) = \mu(A_1) + \mu(A_2) + \ldots.\mu(A_n)$ if $A_1, A_2 \ldots. A_n$ are pairwise disjoint members of $\mathfrak{M}$.

(iii) $A \subset B$ implies $\mu(A) \leq \mu(B)$ if $A \in \mathfrak{M}$, $B \in \mathfrak{M}$.

(iv) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$ and $A_1 \subset A_2 \subset A_3 \subset \ldots$

(v) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$, $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\mu(A_1)$ is finite.

Or

(b) (i) State and prove Fatou's lemma.

(ii) State and prove the Lebesgue's dominated convergence theorem.

10. (a) (i) State and prove the Urysohn's lemma.

(ii) Let X be a locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Prove that λ is regular.

Or

(b) (i) Prove that every set of positive measure has non measurable subsets.

(ii) State and prove the Vitali-Caratheodory theorem.

11. (a) (i) Derive Jeusen's inequality.

(ii) Suppose $1 \le p \le \infty$, and $f \in L^p(\mu)$, $g \in L^p(\mu)$. Prove that $f + g \in L^p(\mu)$ and $\|f + g\|_p \le \|f\|_p + \|g\|_p$.

Or

(b) (i) Let S be the class of all complex, measurable, simple functions on X such that $\mu(\{x : s(x) \neq 0\}) < \infty$. If $1 \le p < \infty$, then prove S is dense in $L^p(\mu)$.

(ii) If X is a locally compact Hausdorff space, then prove that $C_0(X)$ is the completion of $C_c(X)$, relative tot he metric defined by the supremum norm $\|f\| = \sup_{x \in x} |f(x)|$.

12. (a) Let M be a closed subspace of a Hilbert space H. Prove the following.

(i) Every $x \in H$ has then a unique decomposition $x = P_x + Q_x$ into a sum of $P_x \in M$ and $Q_x \in M^{\perp}$.

(ii) P_x and Q_x are the nearest points to x in M and in $M^{\perp}$, respectively.

(iii) The mappings $p : H \to M$ and $Q : H \to M^{\perp}$ are linear.

(iv) $\|x\|^2 = \|P_x\|^2 + \|Q_x\|^2$.

Or

(b) State and prove the parseval theorem.

13. (a) (i) Define the following terms:

(1) Banach algebra;

(2) Invertible element;

(3) Spectrum of an element.

(ii) Derive the spectral radius formula.

Or

(b) (i) Prove that x is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta$.

(ii) Suppose $f(e^{i\theta}) = \sum_{-\infty}^{\infty} c_n e^{i_n\theta}$, $\sum_{-\infty}^{\infty} |c_n| < \infty$ and $f(e^{i\theta}) \neq 0$ for every real θ. Prove that $\frac{1}{f(e^{i\theta})} = \sum_{-\infty}^{\infty} \gamma_n e^{in\theta}$ with $\sum_{-\infty}^{\infty} |\gamma_n| < \infty$.

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