F–3112

Sub. Code
7PMA1C1

M.Phil. DEGREE EXAMINATION, NOVEMBER 2019

First Semester

Mathematics

RESEARCH METHODOLOGY

(CBCS – 2017 onwards)

Time : 3 Hours Maximum : 75 Marks

Part A (5 × 5 = 25)

Answer any **five** questions.

1. Write short notes on :

   (a) Literature survey

   (b) Mathematical model.

2. What are the items under the text of a research report? Explain them in brief.

3. When will you say that an R-module M is said to be cyclic?

   Also prove that an R-module M is cyclic if and only if $M \simeq R/I$ for some ideal I in R.

4. Define the following terms with an example for each :

   (a) Exact sequence

   (b) Projective module.

5. Let I be an ideal with $I \subset \bigcup_{i=1}^{n} P_i, P_i$ prime. Prove that $I \subset P_i$ for some i.

6. State and prove Nakayama lemma.

7. Enumerate the following terms.

   (a) Coercivity

   (b) Lower semicontinuity.

8. State and prove existence of minimizer theorem.

Part B (5 × 10 = 50)

Answer **all** questions choosing either (a) or (b).

9. (a) List and explain the steps of research process.

Or

   (b) Assume a research topic of your choice and give the complete format of its research report.

10. (a) (i) State and prove the Shanuel's lemma.

   (ii) Let $M = \oplus \sum_{i=1}^{n} M_i$ and $N = \oplus \sum_{j=1}^{m} N_j$. Prove that $m \otimes N \simeq \oplus \sum_{i,\, j} (M_i \otimes N_j)$.

Or

(b) For an R-module M, prove the following conditions are equivalent.

(i) A sequence

$O \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow O$ of R-modules is exact if and only if the tensored sequence

$O \longrightarrow M \otimes N' \xrightarrow{f^*} M \otimes N \xrightarrow{g^*} M \otimes N'' \longrightarrow O$ is exact.

(ii) M is R-flat and for any R-module N, $M \underset{R}{\otimes} N = O$ implies $N = O$.

(iii) M is R-flat and for any R-homomorphism $f : N' \to N$,the induced map

$f^* : M \otimes N' \longrightarrow M \otimes N$ is zero implies that $f = 0$.

11. (a) (i) Define the Jacobson radical of R. Give an example.

(ii) Prove that an element $a \in J(R)$ if and only if $1 + ab$ is a unit, for all $b \in R$.

Or

(b) (i) Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence of R-modules. Prove that the induced sequence $M'_S \xrightarrow{fs} M_S \xrightarrow{gs} M''_S$ is exact.

(ii) Prove that R_S is a flat R-module.

(iii) For any two R-modules M and N, prove that $(M \underset{R}{\otimes} N)_S \simeq M_S \underset{R_S}{\otimes} N_S$

12. (a) (i) Discuss the Eler-Lagrange equation.

(ii) Assume L does not depend on Z and the mapping $P \to L(P, x)$ is uniformly convex. Prove that a minimizer $u \in \mathcal{A}$ of $I[0]$ is unique.

Or

(b) (i) State and generalized Dirchlet's principle.

(ii) State and prove weak lower semicontinuity theorem.

13. (a) (i) Narrate the following terms :

Interpretation of the variational inequality, Harmonic mapsand critical point.

(ii) Let $m = n$. Prove that $L(P) = tr(P^2) - tr(P)^2$ $(P \in M^{nm})$ is a null Lagrangian.

Or

(b) State and prove the Defermation theorem.

F–3113

Sub. Code
7PMA1C2

M.Phil. DEGREE EXAMINATION, NOVEMBER 2019

First Semester

Mathematics

FUNCTIONAL ANALYSIS

(CBCS – 2017 onwards)

Time : 3 Hours Maximum : 75 Marks

Section A $(5 \times 5 = 25)$

Answer any **five** questions.

1. Define the following terms :
   (a) Vector space
   (b) Topological space
   (c) Hausdorff space
   (d) Linear mapping.

2. Suppose that $(X_1\, d_1)$ and $(Y_1\, d_2)$ are metric spaces , and $(X_1\, d_1)$ is complete. If E is a closed set in X, $f : E \to Y$ is continuous, and $d_2(f(x'), f(x'') \geq d_1(x', x''))$ for all $x', x'' \in E$, then prove that $f(E)$ is closed.

3. Suppose A is a convex absorbing set in a vector space X. Prove that
   (a) $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$
   (b) $\mu_A(tx) = t\, \mu_A(x)$ if $f \geq 0$.

4. Write the usual notations, prove that L^p is a locally bounded F-space.

5. State and prove the Baire's theorem.

6. Suppose M is a subspace of a vector X, P is a seminorm on X, and f is a linear functional on M such that $|f(x)| \leq p(x)$ $(x \in M)$. Prove that p extends to a linear functional $\wedge$ on X that satisfies $|\wedge x| \leq p(x)\,(x \in X)$.

7. If X and Y are normed spaces and if $\wedge \in \mathfrak{B}(X,Y)$, then prove that $\|\wedge\| = \sup\{|< \wedge x, y^* > 1| : \|x\| \leq 1, \|y^*\| \leq 1\}$.

8. Define the following terms :

   (a) Compact

   (b) Invertible

   (c) Spectrum of an operator

   (d) Eigen value

   (e) Direct sum.

Section B (5 × 10 = 50)

Answer **all** questions, choosing either (a) or (b).

9. (a) Let $\wedge$ be a linear functional on a topological vector space X. Assume $\wedge x \neq 0$ for some $x \in X$. Prove that each of the following four properties implies the other three.

   (i) $\wedge$ is continuous

   (ii) the null space $\mathcal{N}(\wedge)$ is closed

   (iii) $\mathcal{N}(\wedge)$ is not dense in X

   (iv) $\wedge$ is bounded in some neighborhood $\vee$ of 0.

Or

(b) (i) If X is a complex topological vector space and $f : \mathbb{C}^n \to X$ is linear, then prove that f is continuous.

(ii) Show that every locally compact topological vector space X has finite dimension.

10. (a) (i) Define Cauchy sequence. Also prove that sequence $\{x_n\}$ in X is a d-Cauchy sequence if and only if it is a τ - Cauchy sequence.

(ii) Define the following terms.

Bounded linear transformation, Seminorm, Minkouski functional μ_A, Quotient space of X modulo N,Quotient topology.

Or

(b) (i) Prove that a topology vector space X is normable if and only if its origin has a convex bounded neighborhood.

(ii) Show that $C(\Omega)$ is a Frechet space.

11. (a) Suppose

(i) X is an F-space

(ii) Y is a topological vector space,

(iii) $\wedge : X \to Y$ is continuous and linear and

(iv) $\wedge(X)$ is of the second category in Y. Prove that the following:

(1) $\wedge(X) = Y$

(2) $\wedge$ is an open mapping

(3) Y is an F-space.

Or

(b) (i) State the Banach-Steinhaus theorem.

(ii) Define the following terms:

Bilinear mapping and separately continuous.

(iii) State and prove the closed graph theorem.

12. (a) State and prove the Banach-Alaoglu theorem.

Or

(b) State and prove Milman's theorem.

13. (a) If X and Y are Banach spaces and if $T \in \mathcal{B}(X,Y)$, then prove the following three conditions implies the other two

(i) $\mathcal{R}(T)$ is closed in Y

(ii) $\mathcal{R}(T^*)$ is week*-closed in X^*;

(iii) $\mathcal{R}(T^*)$ is norm-closed in X^*.

Or

(b) Suppose X is a Banach space, $T \in \mathcal{B}(X)$, and T is compact. Prove the following :

(i) If $\lambda \neq 0$, then the four numbers

$\alpha = \dim \mathcal{N}(T - \lambda I)$, $\beta = \dim X / \mathcal{R}(T - \lambda I)$ are equal and finite.

(ii) If $\lambda \neq 0$ and $\lambda \in \sigma(T)$ then λ is an eigen value of T and of T^*.

(iii) $\sigma(T)$ is compact, at most countable, and has at most one limit point, namely 0.

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F–3114

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M.Phil. DEGREE EXAMINATION, NOVEMBER 2019

Second Semester

Mathematics

ANALYSIS

(CBCS – 2017 onwards)

Time : 3 Hours Maximum : 75 Marks

Section A $(5 \times 5 = 25)$

Answer any **five** questions.

1. Define the following terms :

   (a) Borel sets of topological space;

   (b) Simple function;

   (c) Measure space.

2. Let $\{E_k\}$ be a sequence of measurable sets in X such that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Prove that almost all $x \in X$ lie in atmost finitely many of the sets E_k.

3. When will you say that a set is said to be compact? Also prove that compact subsets of Hausdorff spaces are closed?

4. Define convex functions. If φ is convex on (a, b), then prove that φ is continuous on (a, b).

5. State and prove the Minkowski inequality.

6. Define the following terms :

   (a) Inner product space;

   (b) Hilbert space;

   (c) Orthonormal set.

7. Derive the Parseval's identity.

8. State and prove the Gelfand – Mazur theorem.

Section B (5 × 10 = 50)

Answer **all** questions, choosing either (a) or (b).

9. (a) State and prove the Labesgue's monotone convergence theorem.

   (b) Support f and $g \in L'(\mu)$ and α and β are complex numbers. Prove that

   (i) $\alpha f + \beta g \in L'(\mu)$;

   (ii) $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f\, d\mu + \beta \int_X g d\mu$.

Or

   (b) (i) Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X such that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. Prove that the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for almost all $x, f \in L'(\mu)$ and $\int_X f\, d\mu = \sum_{n=1}^{\infty} \int_X f_n d\,\mu$.

   (ii) Suppose $f \in L'(\mu)$ and $\left| \int_X f\, d\mu \right| = \int_X |f| d\mu$. Prove that there is a constant α such that $\alpha\, f = |f|$ a.e. on X.

10. (a) State and prove the Riesz representation theorem.

Or

(b) (i) State and prove the Lusin's theorem.

(ii) State and prove the Vitali – Caratheodory theorem.

11. (a) (i) If p and q are conjugate exponents, $1 \leq p \leq \infty$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then prove that $fg \in L^l(\mu)$ and $\|fg\|_l \leq \|f\|_p \|g\|_q$.

(ii) Show that $L^p(\mu)$ is a complete metric space, for $l \leq p \leq \infty$ and for every positive measure μ.

Or

(b) (i) For $l \leq p < \infty$, prove that $C_c(x)$ is dense in $L^p(\mu)$.

(ii) Can you say $C_c(X) \subset C_o(X)$? Why?

(iii) If X is a locally compact Hansdorff space, then prove that $C_o(X)$ is the completion of $C_c(X)$, relative to the metric defined by the superman norm $\|f\| = \sup|f(x)| \quad x \in X$.

12. (a) (i) State and prove the triangle inequality.

(ii) Prove that every nonempty, closed, convex set E in a Hilbert space H contains a unique element of samallest norm.

Or

(b) (i) Show that every orthonormal set B in a Hilbert space H contained in a maximal orthonormal set in H.

(ii) State and prove the Riesz – Fischer theorem.

13. (a) (i) If $x \in A$ and $\|x\| < 1$, then prove that $e + x \in G$,

$(e+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$, and

$\left\| (x+e)^{-1} - e + x \right\| \leq \frac{\|x\|^2}{1 - \|x\|}$.

(ii) For every $x \in A$, prove that $\sigma(x)$ is compact and not empty.

Or

(b) Prove the following :

(i) Every maximal ideal M of A is the Kernel of some $h \in \Delta$.

(ii) $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta$.

(iii) x is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta$.

(iv) $h(x) \in \sigma(x)$ for every $x \in A$ and $h \in \Delta$.

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